# TEERTHANKER MAHAVEER UNIVERSITY
# MORADABAD, INDIA

## CENTRE FOR DISTANCE AND ONLINE EDUCATION

Accredited with NAAC A Grade

12-B Status from UGC

# Programme: Bachelor of Arts (Economics)
# Semester-IV

# Course: Mathematical Methods for Economics

## UNIT – I
## DIFFERENTION AND ITS APPLICATION

**Meaning:**

1. Different ion and the process of finding the rate at which a variable quality in changing
2. The express the rate of change in any function.
3. The concept of derivatives which involves small change in the dependent variable which reference to a small change independent variable.
4. The problem to find a function derivated from the given relationship between two variables so as to express the idea of change.
5. The derived functions is called the derivative of a given function.
6. The process of obtaining the derivative is called differentiation.
7. when a function has derivative it is said to be differentiable.
8. It is denoted by does not mean dy÷ dx
9. It simply means or F (x) the symbol is an operator meaning that differentiation with respect to x.

Rules of derivatives

1. Power Function:

i) The derivatives of a power function y = F(x) = $x^n$

(or) ( ) = $n.x^{n-1}$

= $n.x^{n-1}$

For example :

**1.** If y = $x^5$

$x^n = n.x^{n-1}$

x = x. n = 5

= (5)

= $5.x^{5-1} = 5x^4$

**2.** Differentiate if y = $x^1$

$x^n = n.x^{n-1}$

x = x , n = 1

= ( )

(1)

$= 1.x^{1-1}$

$= 1x^{0}$

$\boxed{y = x^{1}} = 1$

$\boxed{y = x^{1} = 1}$

It is power is zero it will be equal to 1
Equal to 1

**3.** If $y = x^{-7}$

$x^{n} = n.x^{n-1}$

$x = x \quad n = -7$

$= (-7)$

$= -7.\ x^{-7-1}$

$= -7x^{-8}$

**4**. If $y = \frac{1}{7}$

$= x^{-7}$

$x^{n} = n.x^{n-1}$

$= -7.\ x^{-7-1}$

$= -7x^{-8}$

5.(i) . If $y = 2^{1}$ .

$-1$

$= \frac{1}{2}x^{1}$

$= \frac{1}{2\sqrt{}}$

5. (ii). If y = √ ⟹ y = 2

$= n.x^{n-1}$

n = $\frac{1}{2}$

$= \frac{1}{2} \cdot 2^{-1}$

$= \frac{1}{2\sqrt{}}$

$= \frac{1}{2} \times \frac{1}{\sqrt{}} = \frac{1}{2\sqrt{}}$

6. If F (x) = $^{3}_{2}$ find differentiate.

$= n.x^{n-1}$ n = $\frac{3}{2}$

$= \frac{3}{2}$

7. If y = $^{-8}$

$= n.x^{n-1}$
n = $\frac{-8}{3}$

$= \frac{-8}{3} \cdot \frac{-8}{3}$ -1

$= -8 \cdot -3^{11}$

$= -8 \times -3^{11}$

**II. Constant Function:**

I. y = F (x) = c is zero, c is constant

y = c

$\frac{dy}{dx} = \frac{d(c)}{dx} = 0$

**Example**:

1. If y = 25

$\frac{dy}{dx} = \frac{d(25)}{dx} = 0$

2. If y = 1000

$\frac{dy}{dx} = \frac{d(1000)}{dx} = 0$

**II. Derivative of the product of constant**

If y = F(x) = a

Here,
a is the product of constant

$\frac{dy}{dx} = a.n.x^{n-1}$

**Example :**
**1**. If y = $10^{12}$ find $\frac{dy}{dx}$

$\boxed{\frac{dy}{dx} = a.n.x^{n-1}}$

a = 10 , n = 12

$\frac{d(x^{12})}{dx}$ = 10 x 12 x $^{12-1}$

$\boxed{= 120\, x^{11}}$

**2**. If y = a $x^{-4}$ find $\frac{dy}{dx}$

$\frac{dy}{dx}$ = a.n.$x^{n-1}$

a = a , n = -4

$\frac{d(a x^{-4})}{dx}$ = a x -4 x $x^{-4-1}$

$\frac{dy}{dx}$ = - 36$x^{-5}$

**3**. If y = -4 $x^{9}$ find $\frac{dy}{dx}$

$\frac{dy}{dx}$ = a.n.$x^{n-1}$

a = -4 n = 9

$\frac{d(-4x^{9})}{dx}$ = -4 x a x $x^{a-1}$

$\frac{dy}{dx}$ = - 36 $x^{8}$

**4**. If y = -10 $x^{-5}$ find $\frac{dy}{dx}$

$\frac{dy}{dx}$ = a.n.$x^{n-1}$

a = -10 n = - 5

$\frac{d(-10x^{-5})}{dx}$ = -10 x -5 x $x^{-5-1}$

$\boxed{\frac{dy}{dx} = 50\, x^{-6}}$

**III. Linear Function:**

If y = mx + c

Here, m&c are constant

( + ) = m

**Example:**

If y = 5 x +6 find

(5 +6) = 5 x 1 x $^{1-1}$ + 0

= 5 $^{0}$+0

= 5 x (1) = 5

= 5

2 If y = 7x -3 find

(7 −3) = 7 x 1 x $^{1-1}$ + 0

= 7 $^{0}$+0

= 7 x (1) = 7

= 7

**IV. Derivative of a sum**

If y = u + v

= +

**Example**

1. If y = find

= ()+ ()

u = 3

v =

= () = ()

$= n \quad ^{-1}$

$= 3 \; ^{3-1} + 8 \; ^{8-1}$

$= 3 \; ^{2} + 8 \quad ^{7}$

**V. Derivative of a difference:**

If y = u – v

= ()-()

**Example :**

If y = - find

= ()- ()

u =  3

v =  8

= (3)- (8)

= n  −1

= 3 $^{3-1}$-8 $^{8-1}$

=3$^{2}$−8$^{7}$

=3$^{2}$−8$^{7}$

**VI. Derivative of a (product)**

If y = u.v

= u.  + v.

**Example:**

If y = (2 ²+ 9) ( ² + 3 ) find

= u  +

=(2 ²+ 9) (2 ² + 3) + ( ² + 3x) (6 ²+ 3x) (6

² + 6)

= 4 $^4$+ 6 $^3$+ 18x + 27 + 6 $^4$ + 0 + 18 $^3$+ 0

= 10 $^4$ + 24 $^3$ + 18x + 27

**VII. Derivative of the Quotient (Division)**

If y =

Example:

Evaluate for y =

u = x+1 v = x-1

u = x+1

du = 1 $^{1-1}$

= 1x1

= 1+0

du = 1

=1-0

dv = 1

= ( −1)(1) −( +1)(1)

( −1)$^2$

= 1 −1 − 1 +1

( −1)$^2$

−2

( −1)$^2$

= −2

( −1)$^2$

VIII. Exponential Function:

Derivative of an exponent (mLf;Fr;rhh;G)

= n $^{-1}$

1 =1 $^{1-1}$

= $^0$

= 1

1. If y =

2. If y =

3. If y =

4. If y = 8

= 8

IX. Log Funtion (klf;ifr;rhh;G)

1. If y = log x

= (log ) = 1

2. If y = log u

= 1

3. If y = log

= 5 log x

=5(1)

X. Derivative of a composite function (dain rule)
If y is a function of a and u if a function of x
Function of x
(ie) y = F (u) & u = F (x)

1. If y = f (u), u = f (v) and v = f (x) then

= . . and so on

For example:

1. If z = 7y + 3 when y = 5 $^2$ find

Soln:

= .

Z = 7y + 3

Putting y = 5 $^2$

z = 7(5 $^2$) + 3

dz = 7 (10 $^{2-1}$) + 0

= 7 (10x)

= 70x

= 70

2. If y = $^2$ and x = 1 + $^2$, find

= .

= $^{2}{}_{1+}{}^{2}$
= 2 $^{2-1}$.0 + 2 $^{2-1}$
= $2^1.2^1$

= 2x . 2t

Putting x = 1 + $^2$

= 2 (1 + $^2$) . 2t

= (2+2 $^2$) . (2t)

= 4t + 4 $^3$

XI. Derivative of a Parametric Function:

(topayFr; rhh;gpd; tiff;nfO)

If x = f (t), y = g (t)

= ÷ = ×

Ex: 1

If x = a $^3$ and y = 3at find

Soln:

= ÷

= $3a^{1-1} \div 3^{3-1}$

= $3a^{0} \div 3^{2}$

= $3a(1) \div 3^{2}$

= $3a \div 3^{2}$

= 3 ÷ 3 2 = 3 ÷ 1

3 2

Ans:

XII. Derivative of an implicit function (cs;shh;e;jrhh;gpd; tiffSf;F)

Derivative of an explicit function (ntspahh;e;jrhh;gpd; tiffSf;F)

i) The function of the type y = f(x) are called explicit function

ii) The function of the form f(x,y) = c are called implicit function where c is constant.

iii) Here we have to differentiative all the terms both LHS and RHS.

iv) For example y = 2x – 3y -6 is explicit function. 2x – 3 y = 6 is implicit function.

1. find of 2x – 3y = 6

Soln:

= (2 ¹) – (3 ) = (6)

= $2^{1-1} - 3^{1-1} = 0$

= $2^{0} - 3^{0} = 0$

= 2(1) – 3 (1)

= 2 -3 ( is multiplied with RHS = term)

= 2-3 x

= -3 =-2

= = −2 −3

Ans:

= 3 2

XIII. Derivative of an Inverse Funtion:
(jiyfPo;r; rhh;gpd; tiff;nfO)

= 1

Example : 1
Find the derivative of an inverse function y = 2

= n −1
= 2 2−1

= 2x1 = 2x

= 2x

Ans:

1

2

Unit – I

Part – A

1. If y = then is n −1

a) −1 b) n −1 c) n d ) 1

2. The derivative of 2 with respect to x is

a) 3b) x c) 2x d) 2

is

3. If y = 85 then
a) 2 b) 0 c) 85 d) 1

4. If y = 4 3 then is

a) 12 3 b) 12 4 c) 12 2d ) 14 3

$= \frac{1}{2}$ $\frac{-1}{}$

5. If y = √ 1 then

a) b) x c) d)

6. For minimum of a function y = f (x) then $\frac{}{2}$ =

a) Equal to zero b) Greater than zero c) Less than zero d) Equal to one

7. For the total utility function u = 60 $^4$ + 16 $^2$ the marginal utility is

a) 180 $^3$ + 36 $^2$ b) 240 $^3$ + 32x c) 80 $^3$ + 26 $^2$ d) 46 $^3$ + 16 $^2$

8. The Value of if y = $\frac{3}{3}$ is

a) 3 $^2$ b) $^2$ c) $\frac{3}{3}$ d) $^3$

9. For minimum of a function y = f (x)

a) = 1 b) > 0 c) < 0 d) = 0

10. If F (x) = x then f (x) is

a) 0 b) 1 c) x d) –x

11. If y = 2 $^3$ then is

a) 8 $^2$ b)6 x c) 6 $^2$ d) 6) $^3$

12. If y = log x, is

a) $\frac{1}{}$ b) $\frac{1}{2}$ c) x d) $^2$

13. If y = $^7$ then = 7 $^6$

a) 6 b) 5 c) 7 $^6$ d) 3

14. For maximize a function second order condition is equal to

a) 0 b) 1 c) less than zero d) greater than zero

15. If y = 8x + 4 then is

a) 7 b) 8 c) 4 d) 2

16. If y = cos x, is

a) cos x b) sin x c) –cos x d) – sin x

17. If y = $^2$ than , is

a) 2 b) 0 c) $^2$ d) 2 $^2$

18. for maxima is a function y = f(x), =

a) = 0 b) > 0 c) < 0 d) = 1

19. Y = sin x =cos x

20. The value of if y = 3

a) b) c) d)

**High Order Derivatives**

Successive Differentiation

i) is $1^{st}$ order derivative of y

ii) 2 is $2^{nd}$ order derivative it is obtained by differentiating the $1^{st}$ order derivative

iii) 3 is $3^{rd}$ order derivative. It is obtained by differentiating the $2^{nd}$ order derivative and so on.

Example:

If y = 5 $^{4}+2$ $^{3}$ find the $1^{st}$, $2^{nd}$ and $3^{rd}$ order derivative.

Soln:

= n $^{-1}$

Y=5 $^{4}+2$ $^{3}$

$1^{st}$ order derivative

=5 $^{4}+2$ $^{3}$

=20 $^{4-1}+6$ $^{3-1}$

=20 $^{3}+6$ $^{2}$

$2^{nd}$ order derivative

=20 $^{3}+6$ $^{2}$

=60 $^{3-1}+12$ $^{2-1}$

=60 $^{2}+12$

$3^{rd}$ order derivative

=60 $^{2}+12$

=120 +12

= 120 + 12(1)

$3^{\equiv 120}$ + 12

$4^{th}$ order derivative

4

+ 12

$^{\equiv 120}$

$4^{\equiv 120}$

$5^{th}$ order derivative

$5^{= 0}$

Ans:

$1^{st}$ order derivative $= 120^3 + 6^2$

$2^{nd}$ order derivative $^2$ $= 60\ 2 + 12\ 1$

$3^{rd}$ order derivative $= 120 + 12$

$4^{th}$ order derivative $^4$ $= 120 + 12$

$5^{th}$ order derivative $^5$ $= 0$

2. If y = $^2$ find $1^{st}$, $2^{nd}$, $3^{rd}$ order derivative.

Soln:

$= n^{-1}$

Y = $^2$

$1^{st}$ order derivative

$= ^2$

$=2^{\ 2-1}$

$= 2^{\ 1}$

$2^{nd}$ order derivative

$2 \ =2^{\ 1-1}$

$=2 = 2(1) = 2$

$3^{rd}$ order derivated

3

= 0

Ans:

1st order derivative = 2 1

2nd order derivative 2 = 2

3rd order derivative 3 = 0

**Example : 3**

Find the 2nd order derivatives y = (x-1) (2x-1)

Soln:

Y = (x-1) (2 1 − 1)

u = (x-1) (2x-1)

= u. + v.

= (x-1) (2 0- 0) + (2x – 1) (1 0 – 0)

= (x-1) (2) + (2x – 1) (1)

= (2x – 2) + (2x – 1)

= 2x – 2 + 2x – 1

=4 1−3

2 =4 0−0

= 4(1)−0

= 4

Ans:

1st order derivative = 4 1 – 3

2nd order derivative 2 = 4

Example : 4

Find 1st order derivative of y = 1− 2 / 1+ 2

Soln:

$1-{}^2$

Y= $1+2$

=

u = 1 − 2 v = 1 + 2

= $\frac{(1+{}^2)(0-2\ )-(1-{}^2)(0+2\ )}{(1+{}^2)^2}$

= $\frac{(1+{}^2)(-2\ )-(1-{}^2)(2\ )}{(1+{}^2)^2}$

= $(1+{}^2)^2$

= $(1+{}^2)^2$

= $\frac{-4}{(1+{}^2)^2}$

Ans: 1st order derivative = $\frac{-4}{(1+{}^2)^2}$

1. Differentiative y = 6 $^4$ - 7 $^3$ + 3 $^2$ − x + 8 with respect to x

Soln:

= n −1

Y = 6 $^4$ − 7 $^3$ + 3 $^2$ − x +8

= 6 $^4$ − 7 $^3$ + 3 $^2$ − x +8

= 24 $^3$ − 21 $^2$ + 6 $^1$ − 1x +0

= 72 $^2$ − 42 $^1$ + 6x − 1

= 144 $^1$ − 42 x + 6

= 144 x − 42

= 144

Ans :

= 144

2. Differentiate y = +1 with respect to x

$^2$+ +1

Soln:

Y = $\frac{{}^2+\ +1}{{}^2-\ +1}$

$2^{1}+1^{0}+0$

$2^{1}-1^{0}+0$

$2^{0}+1(1)+0$

$2^{0}-1(1)+0$

Y =

Y =

Y = 2(1)+1+0

2(1)−1+0

Y = 2+1+0

2−1+0

X = $1^3$

3. Differentiate y = $3^{4}$ - $2^{3}$ + x + 8 with respect x

Soln:

= n −1

Y = $3^{4} - 2^{3}$ + x + 8

= $12^{3} - 6^{2}$ + 1x + 0

=$36^{2} - 12^{1}$+1

= 72x −12

= 72

Ans: $\boxed{= 72}$

4. Differentiate y = $\frac{3-5}{3+5}$

Soln:

u = 3 – 5 x v = 3 + 5 x

= $\frac{\quad}{\ ^{2}}$

u = 3 – 5x

du = 0 – 5

du = -5

v= 3 + 5x

dv = 0 + 5

dv = 5

= $\frac{\quad}{\ ^{2}}$

— $=\frac{(3+5\ (-5))-(3-5\ (5))}{(3+5)^2}$

— $=\frac{-15-25\ -15+25}{(3+5)^2}$

— $=\frac{-30}{(3+5)^2}$

Ans: = = $\frac{-30}{(3+5)^2}$

5. Differentiate y = ( $^2+1$) (3 $^2-2$)

Soln:

u = ( $^2+1$)    v = (3 $^2-2$)

= $^{-1}$

$^2+1$ = 2x + 0

2 $^{2-1}$=2 $^1$

2 $^{1-1}$=2 $^0$

= 2 (1) =2

(3 $^2-2$)

= 6 $^{1}-0$

= 6 $^{1-1}$

= 6 $^0$

$^0$ = 1

=6(1)

= 6

= .

=(2 $^1$+0)(6 $^1$0)

= (2) (6)

= 18

Ans: — = 18

6. Differentiate y = ( $^2 - 1$) ( $^2 + 2$)

Soln:

u = $^1 - 1$ v = $^1 + 2$

= $^{-1}$

$^2$ - 1 = 2x – 0

2 $^{2-1}$=2 $^1$

2 $^{1-1}$=2 $^0$

=2(1)

= 2

( $^2$+2)

= 2x+0

=2 $^{2-1}$=2 $^1$

=2 $^{1-1}$

= 2 $^0$

= 2(1)

= 2

= . 

=( $^2$–1)( $^2$+2)

=2×2

= 4

Ans:

= 4

7. Differentiate y

= Soln

= . $^{-1}$

n =

1 1 3 3

= =

3 × √ 1

= 3

Ans :

3

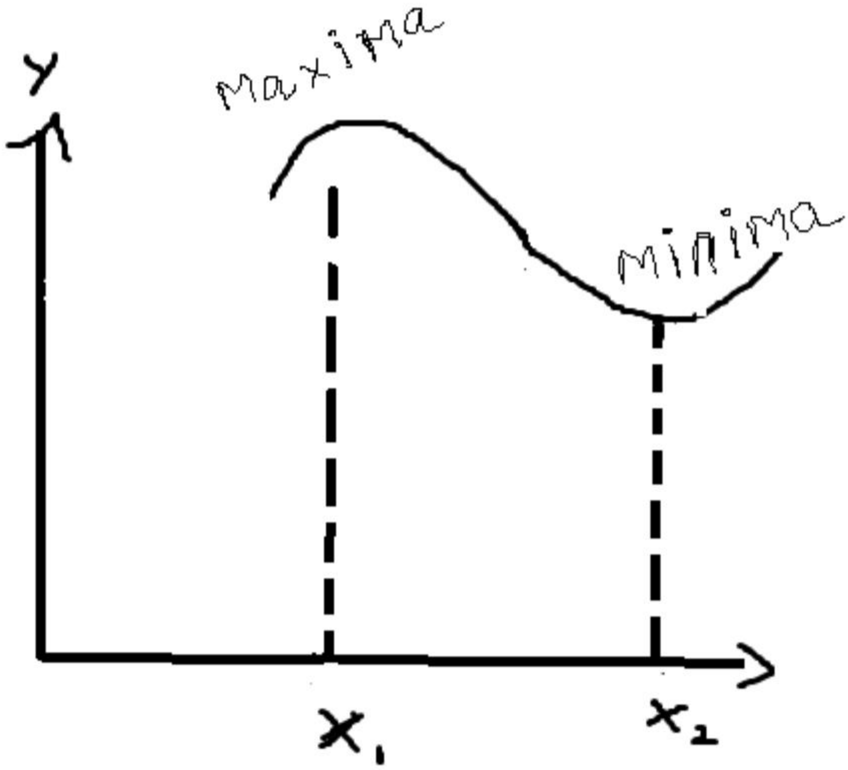
**Order Conditions for Maxima and Minima :**

( kPl;ngLkjpg;Gkw;Wk; kPr;rpWkjpg;G)

| Condition | Maxima | Minima |
|---|---|---|
| $1^{st}$ order | — = 0 | — = 0 |
| II $^{nd}$ order | $< 0$ | $> 0$ |

**Maxima and minima of a function:**

i) for ‘u’ = f ($x_1y$) to have a maximum and minimum a necessary condition is that at their point this will be explain in their following figure.




**Maxima:**

i) In the above figure f (x) is a maximum value (a) of function f. If is highest of the other value of x.

F (x) = x

ii) x is y = f (x)

**Minima:**

i) f ($x_1$) is a minimum value (b) of a function it is the lowest value of

x. Ex:1

Find the maxima and minima of the following function Y=2 $^3$+3 $^2$−36 +10

Soln:

= −1

Y=2 $^3$+3 $^2$− 36 +10

Differentiate with respect to x

=0 ⟹ 6 $^2$+ 6 $^1$− 36 =0–6

= $^2$+ −6=0

= ( x – 2) ( x + 3) = 0

X=-3

X–2=0

X = 2

X=-3,2

Again differentiating

=6 $^2$+6 $^1$− 36

=12 $^1$+6

If x = -3, = 12 (-3) + 6

= -36+6

= - 30

=-30<0

/ function of x = -3 have maximum value, maximum value x = -3 Y=2 $^3$+3 $^2$−36 +10

X=-3

Y = $2(-3)^3 + 3(-3)^2 + 36(-3) + 10$ Y = 2 (-27) + 3 (a) + 08 + 10

Y=-54+27+108+10

Y=91

Maximum = 91

If x = 2

2 =12(2)+6

=24+6

2 30> 0

/ function of x = 2 have minimum value

/minimum value y = $2(2)^3 + 3(2)^2 - 36(2) + 10$ =16+12–72+10

Minimum value = - 34

Ans:

Maximum = 91

Minimum value = - 34

1. Find maxima & minima

Y=2 $^3$–15 $^2$+ 24 −15

Soln :

=

Y=2 $^{3-1}$–15 $^2$+ 24 −1 −0

Differentiate with respect to x

=0 6 $^2$– 30 $^1$+ 24=0

= $^2$–5 $^1$+4=0

= (x -1) (x – 4) = 0

⇨ X–1=0

X = 1

⇨ X–4=0





⇨

X = 4
X=1,4

Again differentiating

$^2$ $=6\,2-30\,1-1+24$

$^2$
$2 = 12x - 30$

If x = 1

$^2$
$2 = 12(1) - 30$
$2 = 12-30$
$2 = -18$
$2 = -18 < 0$

/ function of x = 1 have maximum value

Maximum value x = 1

$Y = 2\,^3 - 15\,^2 + 24 \quad -15$

X = 1

$Y = 2(1)^3 - 15\,(1)^2 + 24(1) - 15$

y = 2 – 15 +24 – 15

y = - 4

maximum = - 4

If x = 4

$^2$
$2 = 12x - 30$
$2 = 12\,(4) - 30$
$2 = 48-30$
$2 = 18$
$2 = 18 > 0$

/ function of x = 1 have minimum value

/minimum value y = 2 $^{3}$ – 15 $^{2}$ + 24 – 15

y = $2(4)^3 - 15(4)^2 + 24(4) - 15$

y = 2(64) – 15 (16) + 24(4) -15

y = 128 – 240 +96 -15

y = -31

Minimum = -

31 Ans :

Maximum = -4

Minimum = -31

## Part –B

1. If y = 5 $^{3}$ – 6 $^{2}$ + 10 + 10 find and $^{2}$

2. If y = (5 $^{2}$) (3 $^{2}$ + 3) then find

3. If y = 5 $^{4}$ + 2 $^{3}$find , $^{2}$ and $^{3}$

4. If y = (2 $^{3}$+9) ( $^{2}$+3x) find

5. Find the third order derivative of the function y = 60 $^{2}$+ 12x

6. Explain the rule of differentiation with example

7. If y = 4 $^{5}$ + 3 + 5 find the first, second and third derivative.

8. find for y = ( $^{2}$ – 1) ( $^{2}$ + 2)

9. find the 2$^{nd}$ order derivatives of y = $^{7}$ + 7 $^{3}$ – 3 $^{2}$ + 15

10. find if y = +1

11. find for y = −1 $^{3}$ + 4 $^{4}$ + 8 + 2

12. find the 2$^{nd}$ order derivatives of y = (x-1) (2x-1)

13. If y = ( 4 + 3) ( 2 + ) find

14. If y = $\frac{^{2}-1}{+1}$ then find

## Part - C

1. If = (2 3 + ) ( 2 + 3 ) find

2. If y = 4 $^{5}$ + 3 + 5 find 1$^{st}$ , 2$^{nd}$& 3$^{rd}$derivating.

3. find , if i) y = $(2\,3 +\quad)(2 + 3)$

$-1$

ii) y =

# 4.find if i) y =

ii) y = √

iii) y = $^{-7}$

iv) y = 12

5. find maxima and minima of the following function Z=48–4 $^2-2\,^2+16+12$

6. Find the maxima and minima of the following function Y=2 $^3-3\,^2-36+10$

7. Determine the maxima and minima

Y= $^4-3\,^3+3\,^2-$

8. Find the maxima and minima of a function

Y= $^3-5\,^2+3+5$

9. If y = $3+2$ then find

10. Determine the maxima and minima of the function

Y= $^3-2\,^2+\;+4$

11. Find the first order derivatives of y = 1+ :

1. If y = $5\,3-6\,2+10\;+10$ find and $^2$

Soln:

Given:

Y=5 $^3-6\,^2+10\qquad +10$

Find:

and $^2$

Soln:

1$^{st}$ order derivative

=5 $^3-6\,^2+10+10$

$= 3 \times 5 \times \quad ^{3-1} - 2 \times 6 \times \quad ^{2-1} + 1 \times 10 \times \quad ^{1-1} + 0$

$= 15\ ^{2} - 12\ ^{1} + 10\ ^{0} + 0$

$= 15\ ^{2} - 12\ ^{1} + 10 + 0$

$= 15\ ^{2} - 12\ + 10$

2nd order derivative

$\frac{2}{2} = 15\ ^{2} - 12\ + 10$

$\frac{2}{2} = 2 \times 15 \times \quad ^{2-1} - 1 \times 12 \times \quad ^{1-1} + 10$

$\frac{2}{2} = 30\ ^{1} - 12\ ^{0} + 0$

$\frac{2}{2} = 30\ ^{1} - 12(1) + 0$

$\frac{2}{2} = 30x - 12$

Ans:

1st order derivative

$= 15\ ^{2} - 12\ + 10$

2nd order derivative

$\frac{2}{2} = 30x - 12$

3. If $y = 5\ ^{4} + 2\ ^{3}$ find $\frac{}{}$, $\frac{^2}{2}$ and $\frac{^3}{3}$

Given:

$Y = 5\ ^{4} + 2\ ^{3}$

Find:

$\frac{}{}$, $\frac{^2}{2}$ and $\frac{^3}{3}$

Soln:

$= \quad ^{-1}$

1st order derivative

$= 5\ ^{4} + 2\ ^{3}$

$= 4 \times 5 \times \quad ^{4-1} + 3 \times 2 \times \quad ^{3-1}$

$= 20\ ^{3} + 6\ ^{2}$

2nd order derivative

=20 3+6 2

= 3x 20 x 3−1 + 2 x 6 x 2−1

=60 2+12 1

3rd order derivative

=60 2+12 1

= 2 x 60 x 2−1 + 1 x 12 x 1−1

=120 1+12 0

= 120 x +12(1)

3 120 x +12

Ans:

1st order derivative

=20 3+6 2

2nd order derivative

=60 2+12 1

3rd order derivative

3 = 120 x +12

6. Find the maxima and minima of the following function Y=2 $^3$−3 $^2$−36 +10

Given :

Y=2 $^3$−3 $^2$− 36 +10

Find:

Maxima and minima

Soln:

−1

Y=2 $^3$−3 $^2$− 36 +10

Differentiate with respect to x

$\Rightarrow$

$= 06^2 - 6\ ^1 - 36 = 0 \div 6$

$= 0^2 - 6 = 0$

= (x – 3) (x +1) = 0

$\Rightarrow$ x – 3 = 0

x = 3

x + 2 = 0

x = -2

x = 3 – 2

Again differentiating

$=6\ \ 2 - 6\ 1 - 36$

$=12\ \ 1 - 6$

If x = 3

$= 12\ (3) - 6$

$= 36 - 6$

$= 30 > 0$

/ function of x = 3 have minimum

value Minimum value x = 3

$Y = 2\ ^3 - 3\ ^2 - 36\ + 10$

X = 3

$Y = 2(3)^3 - 3\ (3)^2 - 36\ (3) + 10$

Y=2(27)–3(9)–36(3)+10

Y=54-27–108+10

Y=-71

Minimum = -71

If x = -2

$= 12x - 6$

$= 12(-2) - 6$

2

$2^{=-30<0}$

/ function of x = -2 have maximum value /maximum value y = $2^{3} - 3^{2} - 36 + 10$

Y = $2(-2)^3 - 3(-2)^2 - 36(-2) + 10$

Y=2(-8)–3(4)–36(-2)+10

Y=-16–12+72+10

Y=54

Maximum = 54

Ans:

Minimum = - 71

Maximum = 54

Part – B

7. If y = $4^{5}$ + 3 x + 5 find the first, second, and third derivatives. Given:

Y = $4^{5}$ + 3 x + 5 Find:

First, second & third derivatives

Soln:

= $^{-1}$

$1^{st}$ order derivative

=4 $^{5+3\ 1+5}$

= 5 x 4 x $^{5-1} + 3\ ^{1-1} + 0$

=20 $^{4}+3\ ^{0}$

=20 $^{4}+3(1)$

=20 $^{4}+3$

$2^{nd}$ order derivative

$\frac{2}{2}$ =20 $^{+3}$

$2 = 4 \times 20 \times \quad 4-1 + 3$

$=80$

$3^{rd}$ order derivative

$=80$

$= 3 \times 80$

$=240$

Ans:

$1^{st}$ order derivative $= 20 \quad 4+3$

$2^{nd}$ order derivative $2 \quad = 80 \; 3 + 0$

$3^{rd}$ order derivative $= 240$

Part – C

3. Find , if i) $y = (2 \quad 3+9)(2+3)$ ii) $y = \frac{-1}{2+1}$

i) $y = (2 \quad 3+9)(2+3)$

u v

given:

$y = (2 \quad 3+9)(2+3)$

find :

u. — + v.—

soln:

$u = (2 \quad 3+9)$ du $= 6\,2 + 0$ $v = (2+3)$ dv = 2x + 3

$=(2\,3+9)$ $(2x+3) + (2+3)(6\,2+0)$

$= (6\,2+0)(2+0) + (2x+3)(12x)$

$= (12x)(0) + (2+0)(12)$

$= 0 + (2)(12)$

$= 24$

Ans: $\boxed{\quad = 24}$

ii) $y = \frac{-1}{2+1}$

soln:

$y = \frac{-1}{2+1}$

u = x -1 $\quad v = {}^{2} + 1$

$= \frac{\quad}{^{2}}$

u = x – 1 $\quad du = 1\,{}^{0} - 0$

$v = {}^{2} + 1 dv = 2\,{}^{1} - 0$

$= \frac{({}^{2}+1)(1\,{}^{0}-0)-(x-1)(2\,{}^{1}-0)}{({}^{2}+1)^2}$

$= \frac{({}^{2}+1)(1(1))-(x-1)(2\,{}^{1-1})}{({}^{2}+1)^2}$

$= \frac{({}^{2}+1)(1)-(x-1)(2\,{}^{0})}{({}^{2}+1)^2}$

$= \frac{({}^{2}+1)-(x-1)(2(1))}{({}^{2}+1)^2}$

$= \frac{({}^{2}+1)-(x-1)(2)}{({}^{2}+1)^2}$

$= \frac{({}^{2}+1)-(2x-2)}{({}^{2}+1)^2}$

$= \frac{2\,{}^{1-1}+0-2\;-2}{({}^{2}+1)^2}$

$= \frac{2\,{}^{0}+0-2\;-2}{({}^{2}+1)^2}$

$= \frac{2(1)+0-2(1)-2}{({}^{2}+1)^2}$

$= \frac{2-2-2}{({}^{2}+1)^2}$

$= \frac{-2}{(\quad+1)}$

Ans: $\boxed{= \frac{-2}{({}^{2}+1)^2}}$

**Part – B**

2. If y = (5 1)(3 2 + 3) then find

Given:
Y=(5 2)(3 2+3)
Find :

—

Soln:
Y=(5 2)(3 2+3)

— = u. — + v. —

u = 5 2 v = 3 2 + 3

= . −1

— = c = 0

=(5 2)(2×3× 2−1+0)+(3 2+3)(2×5× 2−1)
= (5 2) (6 1) + (3 2 + 3) (10x)
= 30 3+ 30 2 + 30x
= 30 3+ 30 2 + 30x

Ans:
4. If y = (2 3 + 9) ( 2 + 3 ) find

Soln:

= u. + v.
u = 2 3 + 9 — v = 2 + 3
=(2 3+9)(2 2−1+3 1−1)+ 2+3 )(3×2× 3−1+0)
= (2 3 + 9) (2x+3 0) + ( 2 + 3 ) (6 2)
= (2 3 + 9) (2x+3(1)) + ( 2 + 3 ) (6 2)
= (2 3 + 9) (2x+3) + ( 2 + 3 ) (6 2)
=4 4 + 6 3 + 18x+27 + 6 4 + 18 3

—

Ans: — =16 4 + 24 3 + 18x+27

5. Find 3rd order y = 60 2 + 12x Soln:

= 60 2 + 12x

= 2 × 60 × 2−1 + 1×12× 1−1

=120 1+12 0

= 120 x + 12 (1)

= 120 x + 12

2

2 = 120 x + 12

=120 0

2 = 120 (1)

2 = 120

3 = 120

3 = 0

Ans: 3 = 0

8. Find for y = ( 2 − 1 ) ( 2 + 1 )

Soln:

Y=( 2−1)( 2+1)

u = ( 2 − 1 ) v = ( 2 + 1 )

= u. + v.

$=(\ ^{2}-1)\ (2\times\ ^{2-1}+0)+(\ ^{2}+2)\ (2\times\ ^{2-1}-1)$

$=(\ ^{2}-1)\ (2x)+(\ ^{2}+2)\ (2x)$

$=2\ ^{3}-2x+2\ ^{3}+4x$

$=4\ ^{3}-2x$

$=4\ ^{3}-2x$

Ans :

9. Find the 2nd order derivative y = $\ ^{7}+7\ ^{3}-3\ ^{2}+15$

Soln:

Y= $\ ^{7}+7\ ^{3}-3\ ^{2}+15$

$=7\times\ ^{7-1}+3\times7\times\ ^{3-1}-2\times3\times\ ^{2-1}+0$

$=7^{6}+21\ ^{2}-6$

$^{2}=7\ 6+21\ 2-6$

$2$

$^{2}=6\times7\times\ _{6-1}+2\times21\times\ _{2-1}-1\times6\ _{1-1}$

$2$

$^{2}=42\ _{5}+42\ -6_{0}$

$2$

$^{2}=42\ _{5}+42\ -6(1)$

$2$

$^{2}=42\ _{5}+42\ -6$

$2$

Ans: $2$

$_{2}=42\ ^{5}+42\ -6$

10. Find $\overline{\ \ }$ if y = $\frac{+1}{-1}$

Soln:

Y = $\frac{+1}{-1}$

$= \frac{\ \ \ \ \ \ \ \ }{2}$

u = (x – 1) v = (x+1)

$=\frac{(+1)(1\times\ ^{1-1}-0)-(\ -1)(1\times\ ^{1-1}+0)}{(+1)^{2}}$

$\frac{}{} = \frac{(+1)(1\times{}^{0}) - (-1)(1\times{}^{0})}{(+1)^2}$

$\frac{}{} = \frac{(+1)(1\times1) - (-1)(1\times1)}{(+1)^2}$

$\frac{}{} = \frac{(+1)(1) - (-1)(1)}{(+1)^2}$

$\frac{}{} = \frac{(1+1) - (1-1)}{(+1)^2}$

$\frac{}{} = \frac{+1 - -1}{(+1)^2}$

$\frac{}{} = \frac{2}{(+1)^2}$

Ans: $\boxed{\frac{}{} = \frac{2}{(+1)^2}}$

11. Find for y = $^3 + 4\,^4 + 8 + 2$

Soln:

y = $^3 + 4\,^4 + 8 + 2$

$=3\times{}^{3-1} + 4\times4\times{}^{4-1} + 1\times8\times{}^{1-1} + 0$

$=3\,^2 + 16\,^3 + 8\,^0 + 0$

$=3\,^2 + 16\,^3 + 8(1) + 0$

$=3\,^2 + 16\,^3 + 8 + 0$

$=3^2 + 16\,^3 + 8$

Ans: $=3^2 + 16\,^3 + 8$

12. Find 2$^{nd}$order, derivative y = (x – 1) (2x -

1) Y = (x – 1) (2x -1)

Soln:

$\frac{}{} = u.\frac{}{} + v.\frac{}{}$

$= (x-1)(1\times 2\times{}^{1-1} - 0) + (2 - 1)(1\times{}^{1-1} - 0)$

$= (x-1)\ (2\ ^{0} - 0) + (2\ \ - 1)\ (\ ^{0} - 0)$

$= (x-1)\ (2(1)\ \ - 0) + (2\ \ - 1)\ (1\ \ - 0)$

$= (x-1)\ (2) + (2 - 1)\ (1)$

$=(2\ -2)+(2\ -1)$

$=2\ -2+2\ -1$

$=4\ -3$

$=4\ -3$

Ans:

13. If y =( $^{4}$- $^{3}$) ( $^{2}$+x) find

Soln:

= u. + v.

u = ( $^{4}$- $^{3}$) v = ( $^{2}$+x)

= ( $^{4}$- $^{3}$) (2× $^{2-1}$ + 1) + ( $^{2}$+x) (4× $^{4-1}$- 3 × $^{3-1}$)

= ( $^{4}$- $^{3}$) (2 $^{1}$ + 1) + ( $^{2}$+x) (4 $^{3}$- 3 $^{2}$)

= ( $^{4}$- $^{3}$) (1×2× $^{1-1}$ + 0) + (2 × $^{2-1}$+1)

# UNIT – II
## PARTIAL DERIVATIVES

If u = xy, u = x+y, u = xy + yx

Ex: 1

If u = xyfind — and —

Given:
U = xy

Find:
__and__

Soln:

=
/ = n $^{-1}$
= 1 $^{1-1}$
= $^{0}$

= 1y
= 1
/ = n $^{-1}$

=
= x 1 $^{1-1}$
= x $^{0}$

= x1

=

2. If u = $^{2}$ y find — & —

Given:

Find:

&

Soln:

$= n^{-1}$

$= 2\ 2-1$

$= 2\ 1$

$= 2$

2y

$= 2_1\ 1-1$

2 0

$= 2-1$

$= 2$

3. If u = $^2+$ y find &

Given:

u = $^2$+y

find :

&

Soln:

$= n^{-1}$

$=$ 2

$= 2\ 2-1$

$= 2\ 1$

$= 2$

$=1\ 1-1$

$=1\,0$

$=1\times 1$

$= 1$

4. If x+yfind &

Given:

x+y

find:

&

Soln:

$=1\ 1-1$

$= 1\,0$

$=1\times 1=1$

$= 1$

$=1\ 1-1$

$=1\,0$

$=1\times 1$

$= 1$

**High Order Partial Derivatives**

1. If u = $^3 + 3\ ^2 + ^3$ for all the higher order partial derivatives. Given:

u = $^3 + 3\ ^2 +\ ^3$

Find:

High order partial derivative

Soln:

= n $^{-1}$

= 3× $^{3-1}$+ 2× 3 × $^{3-1}$y

= 3 $^2$+ 6 $^1$y

= 2×3× $^{2-1}$+ 1×6× $^{1-1}$y

= 6 $^1$+ 6 $^0$y

= 6x + 6(1)y

= 6x + 6y

5.find &

Z= $^2+^2+$ + +

Given:

Z= $^2+^2+$ + +

Find:

&

Soln:

= n $^{-1}$

=2× $^{2-1}$+1 $^{1-1}$ + +

=2$^1$+1$^0$ +

=2 $^0$+1(1)

= 2(1)+1 + 1

= 2+1

$= \quad ^2 + ^2 + \quad + \quad +$

$=0+2 \quad ^1 + 1\ ^0 + 0 + 1\ ^0$

$= 0+2\,y + 1(1) + 0 + 1(1)$

$= 0+2\,y + 1 + 0 + 1$

**Part - A**

1. If u = xy then — is

a) x b) y c)1 d)0

2. If Q = 48KL - 20 $^2$ then __ is

a) 48K – 40L b) 48L – 20 $^2$ c) 48K - 20 $^2$ d) 96L – 40L

3. If the production function Q = 24KL - 10 $^2$ then marginal productivity of labour is equal to

a) 24L b) 24K – 20L c) 24K - 10 $^2$ d) 24KL – 20L

4. If z = $^2$+ $^2$ then __value is

a) 2x b) 2y c) $^2$ d) $^2$

Q=24KL-10 $^2$

= 24K1×

=24K $^0$–20 $^1$

= 24K (1) – 20L

= 24K– 20L

5. u = $^3$ then value is

a) $^3$ b) $^2$ c) 3x d) 9x

6. If u = then __ is

a) b) x. c) y. d) 1

7. If u = 6 - 3 -7 then —

a) 6x b) 12 c) 12x d) 6

8. If u = 3x – 9y +2 then —

a) 3 b) 9 c) -3 d) -9

9. If u = – , — = — ___ -

a) b) c) x d)

10. If u =, __ =

a) e 2 2 b) — c) — d)

**Higher Order Partial Derivative**

1. If u (x,y) = 1000 - 3- 2+ 4 3 6+ 8y find the higher order partial derivatives.

Soln:

u (x,y) =1000 - 3- 2+ 4 3 6+ 8y

=0-3× 3−1-0+3×4 3−1 6+0

=-3 2+12 2 6

=-3 2+12 2 6

=0–0-2 2−1+4 3 6 6−1 1−1 )+8

=–2 1+4 3×6 5+8 0

=–2 1+4 3×6 5+8(1)

= –2 y+24 3 5+ 8

2 = ( ) -3 2+12 2 6

=2×-3 1−1+2×12× 2−1 6

2 =-6 1+24 1 6

2 ( ) (- 2y+24 3 5+8)

$^2$ =-6+24 $3(5\times 5-1)+0$

2 = 2+24 3(5 4)+0

2

=2+124 3 4

2

= ( ) ⇒ (- 2y+24 $^3$ $^5$+ 8)

= $72^{25}$

2

=72

2 = ( )

(-3 $^2$+12 $^2$ $^6$)

= 12 $^2$(6 $^5$)

2

=72

1. If z = 2 $^3$ + 5 $^2$ + $^2$+ $^2$ verify that

=

Soln:

Z=2$^3$+5$^2$

=3×2× $^{3-1}$+2×5× $^{2-1}$( )+1( $^2$)+0

=6 $^2$+10 ( )+ 2

=6 $^2$+10 + $^2$

=0+5 $^2$(1)+ (2 )+2 $^1$

$= 5^2 + 2 + 2y$

$^2 = (\ ) = (6^2 + 10 + ^2)$

$= (2 \times 6 \times \quad ^{2-1} + 10 \times 1 \times ^{1-1}() + ^2)$

= (12x + 10(y) + 0 )

$^2_2 = 12x + 10y$

$^2 = (\ ) = \ = (5^2 + 2 + 2)$

$= 0 + (2 \times 1 \times ^{1-1} + 2)$
$= 0 + (2 \times 1) + 2$

$= 2 + 2$

$^2 = (\ ) = (5^2 + 2 + 2)$

$= 2 \times 5 \times ^{2-1} + 1 \times ^{1-1}(2y) + 0$
$= 10^1 + 1(2y) + 0$

= 10x + 2y

$^2 = \quad = 6^2 + 10 + ^2)$

$= 0 + 10x(1) + 2 \times ^2$
$= 10x + 2^1$

Ans:

$^2 = ^2$

Find the partial deviation of z = 4 $^2$ + 4 + $^2$

Soln:

Z=4 $^2$+4 + $^2$

=2×4× $^{2-1}$+1×4× $^{1-1}$ +0

=8 $^1$+4 $^0$( )+0

= 8x+4 (1) + 0

= 8x+4 (1) + 0

= 8x+4

=0+4 $^1$× $^{1-1}$+2× $^{2-1}$

=0+4 $^0$+2 $^1$

=0+4 (1)+2

=4 +2

2

2 = 8x + 4y

2

2 = 1×8x + 4y

2 = 8 + 4y

= 8(1)+ 0

2

2 = 8

= 4x + 2 0

2

2 = 4x + 2(1)

2=0+2

2 = 2

2 = 8x + 4y

2 2 = 0 + 4y

= 4y

2 = 4x + 2y

2 =4 0+0

2 2 = 4(1)

= 4

------1

Find Maxima and Minima

**1.** z = 48 − 4 $^{2}$ − 2 $^{2}$ + 16x + 12y

**2.** y = 2 $^{3}$- 3 $^{2}$- 36x + 10

**3.** y = $^{4}$- 3 $^{3}$+ 3 $^{2}$- x

**4.** y = $^{3}$+ 5 $^{2}$+ 8x + 5

**5.** y = $^{3}$ − 2 $^{2}$+ x + 4

**4.** y = $^{3}$+ 5 $^{2}$+ 8x + 5

Soln:

= = $^{3}$+ 5 $^{2}$+ 8x + 5

=3× $^{3-1}$+ 2×5× $^{2-1}$+ 1×8× + 0

=3 $^{2}$+10 $^{1}$+8 $^{0}$+0

$=3\ ^{2}+10\ ^{1}+8(1)$

$=3\ ^{2}+10\ ^{1}+8$

$=(x+2)(3x+4)$

X+2=0

X=-2

3x + 4 = 0

3x = -4

X =

X=-2,

Again differentiating.

$=3^{2}+10x+8$

$=2\times3^{2-1}+1\times10\times^{1-1}+0$

$=6^{1}+10^{0}$

$=6x+10(1)$

$=6x+10$

If x = -2

$=6(-2)+10$

$=-12+10$

=-2

=-2<0

/ function of x = -2 have maximum value

$^{3}+5\ ^{2}+8x+5$

/maximum value $(-2)^{3}+5(-2)^{2}+8(-2)+5$

=-8+20–16+5

Maximum value = 1

$-4$ $2$ $-4$

+ 10

If x = $_3$ , 2 = 6 ×

=-8+10

= 2

$2$ =2>0

/ function of x = 2 have minimum value Y = $^3$+ 5 $^2$ + 8x + 5

= $2^3$+ $5(2)^2$ + 8(2) + 5

=8+20+16+5 = 49

Minimum Value = 49

**2**. y = $^3 - 2\,^2$ + x + 4

Soln:

=3× $^{3-1}$−2×2× $^{2-1}$+1

=3 $^2$−4 $^1$+1

x + $^{-1}$ , x + - 1

x - $^{-1}$ = 0 x -1 = 0

x = $^1$, x = 1

Again differentiating.

=3$^2$ − 4x + 1

=2×3× $^{2-1}$ − 1×4× $^{1-1}$ + 0

=6$^1$−4$^0$

2

2

=6x – 4

If x = 1

2 =6( 3 )–4

2 =6( 3 )–4

2 =2–4

2 =-2

2 =-2<0

/ function of x = - 2 have maximum value

3 – 2 2 + x + 4

/Maximum value x = - 2

$(-2)^3 - 2(-2)^2$ + (-2) + 4

- 8–8–2+4 =-

14

If x = 1

= 6x – 4

= 6(1) – 4

= 6 – 4

= 2

= 2>0

/ function of x = 2 have minimum value

Y = $^3 - 2\ ^2 + x + 4$

$Y = 2^3 - 2(2)^2 + 2 + 4$

Y=8–8+2+4

Y = 6

Minimum Value = 6

$\frac{3-5}{3+5}$

Revise:

1. Differentiate y =

Soln:

Y = $\frac{3-5}{3+5}$

u = 3 – 5x  v = 3 + 5x

$\frac{}{} = \frac{\cdot\ - \cdot}{^2}$

du = 3 – 5x

du = 0 – 1 × 5 × $^{1-1}$

du = 0 – 5 $^0$

du = -5 (1)

du = -5

dv = 3+ 5x

dv = 0+1×5× $^{1-1}$

dv = 0+ 5 $^0$

dv = 0+ 5(1)

dv = 5

$\frac{}{} = \frac{3+5\ (-5) - 3-5\ (5)}{(3+5\ )^2}$

$\frac{}{} = \frac{-30}{(3+5\ )^2}$

2. Differentiate y = 6 $^4$ –7 $^3$ + 3 $^2$– x + 8 with respect to x.

Soln:

$= n\ ^{-1}$

Y = = 6 $^4$ – 7 $^3$ + 3 $^2$ – x + 8

=4×6× $^{4-1}$–3×7× $^{3-1}$+2×3× $^{2-1}$–1× $^{1-1}$+0

= 24 $^3$ – 21 $^2$+ 6x– $^0$

=3×24× $^{3-1}$–2×21× $^{2-1}$+1×6× $^{1-1}$–1

$=72 \quad {}^{2} - 42\,{}^{1} + 6\,{}^{0}\ 0$

$=72 \quad {}^{2} - 42 + 6(1)$

$=72 \quad {}^{2} - 42 + 6$

$= 2 \times 72 \times \quad {}^{2-1} - 1 \times 42 \times {}^{1-1} + 0$

$=144 \quad {}^{1} - 42\,{}^{0}$

$= 144\ x - 42(1)$

$= 144\ x - 42$

If y = $(5\ {}^{2})\ (3\ {}^{1} + 3)$ then find

Given:

Y = $(5\ {}^{2})(3\ {}^{1}+3)$ Find:

Soln:

$= u. \qquad + v.$

$u = 5\ {}^{2}$

$du = 2 \times 5 \times {}^{2-1}$

$du = 10\ {}^{1}$

$v = (3\ {}^{2}+3)$

$dv = 2 \times 3 \times {}^{2-1} + 0$

$dv = 6x$

$= (5\ {}^{2}) . (6x) + (3\ {}^{1} + 3).(10x)$

$= 30 \quad {}^{3} + 30\ {}^{3} + 30x$

$=60 \quad {}^{3} + 30$

If $z = 2 \quad {}^{3} + 5\ {}^{2}y + x\ {}^{2} + {}^{2}$ verify that

$\frac{{}^{2}}{\quad} = \frac{{}^{2}}{\quad}$

Soln:

$Z = 2 \quad {}^{3} + 5\ {}^{2}y + x\ {}^{2} + {}^{2}$

$=3 \times 2 \times {}^{3-1} + 2 \times 5 \times {}^{2-1}y + 1 \times x \qquad {}^{2+0}$

$=6\ {}^{2} + 10\ {}^{1}y + \qquad 2$

$=6\ ^2 + 10xy + \qquad ^2$

$=0 + 5\ ^2(1) + x\ 2\times \qquad ^{2-1} + 2\times\ ^{2-1}$

$=0 + 5\ ^2 + x\ 2\ ^1 + 2\ ^1$

$=5\ ^2 + x\ 2y + 2y$

2

=

2

$= 2 \times 6 \times\ ^{2-1} + 1\times 10\ ^{1-1} y + 0$

$= 12\ ^1 + 10\ ^0 y + 0$

$= 12x + 10(1)y + 0$

$= 12x + 10y$

2

2 $= 12x + 10y$

2

=

2

$=0+ \qquad 1\times 2\times\ ^{1-1} + 1\times 2\times\ ^{1-1}$

$=0+ \qquad 2\ ^0 + 2\ ^0$

$= 0 + x2(1) + 2(1)$

$= x2 + 2$

2

$= (\quad ) = 5\ ^2 + x2y + 2y$

$=2\times 5\times\ ^{2-1} + 1\times\ ^{1-1} 2 \qquad +0$

= 10 $^{1}$+ $^{0}$(2 )

= 10x + 1 (2y)

2 = 10x + 2y

2 ( 6 $^{2}$ + 10xy + $^{2}$

= ) =

= 0 + 10x1× $^{1-1}$+ 2× $^{2-1}$

= 0 + 10x $^{0}$+ 2 $^{1}$

= 0 + 10x(1) + 2y

2

=10x + 2y

Ans:

= 2

**Maxima and Minima:**

3. y =

Soln:

a= 4 b = -9 c = 6

x= $\frac{-(-9) \pm \sqrt{(-9)^2 - 4(4)(6)}}{2(4)}$

x= $\frac{9 \pm \sqrt{81-96}}{8}$

x= $\frac{9 + \sqrt{-15}}{8}$

x= $\frac{9 \pm -7.5}{8}$

x= $\frac{9+(-3.8)}{8}$

x= $\frac{12.87}{8}$

x=

x = 1.608

x = 1.608

$\frac{9-3.87}{8}$

$\frac{5.13}{8}$

x=

x=

x = 0.64

$\boxed{x = 0.64}$

Partial Derivatives of

$z = 4\ {}^{2} + 4xy + \quad {}^{2}$

given:

$z = 4\ {}^{2} + 4xy + \quad {}^{2}$

find:

Partial derivative

Soln:

$= 4\ {}^{2} + 4x + \quad {}^{2}$

$= 4\ 2-1 + 4\ 1-1 + 2$

$= 2 \times 4 \times \quad {}^{2-1} + 1 \times 4 \times {}^{1-1} + 0$

$= 8\ {}^{1} + 4\ {}^{0}$

$= 8\ {}^{1} + 4(1)$

$= 8x + 4$

$= 4\ {}^{2} + 4x\ {}^{1} + \quad {}^{2}$

$= 0 + 4 \quad {}^{0} + 2\ {}^{1}$

$= 4\ (1) + 2\ {}^{1}$

$= 4 + 2y$

3. Find the Second order derivative of y = (x-1) (2x-1)

Given:

y = (x-1) (2x-1)

Find:
Second order derivative
Soln:
y = (x-1) (2x-1)
u = (x-1) (2x-1)

= u. __ + v. __

u = (x-1) du = (x-1)
du = 1× $^{1-1}$- 0
du = 1 $^{0}$
du = 1 (1)
du = 1

v = (2x-1)
dv = 1× 2 × $^{1-1}$- 0
dv = 2 $^{0}$
dv = 2 (1)
dv = 2

= u. + v.

= (x-1) (2) + (2x-1) (1)

=2x – 2 + 2x – 1

=4x – 3

Second order derivative
2
2 = 4x – 3
2
2 =1×4× $^{1-1}$–0
2
=4$^{0}$
2
2 = 4(1)
2
2 = 4
Ans:
Second order derivative $\frac{2}{2 = 4}$

4. z = 2 $^3$ + 5 $^2$ + $^2$ + $^2$ verify that

$\frac{^2}{} = \frac{^2}{}$

Soln:

z = 2 $^3$ + 5 $^2$ + $^2$ + $^2$

=3×2× $^{3-1}$+2×5× $^{2-1}$ +1× $^{1-1}$ +0

=6$^2$+10$^1$ +1$^0$$^2$

=6 $^2$+10 +1(1) $^2$

$\boxed{= 6\,^2 + 10 \;+\,^2}$ -----①

=2$^3$+5$^2$ + $^2$+$^2$

=0+5 $^2$× $^{1-1}$+ 2× $^{2-1}$+2 $^{2-1}$

=5$^{20}$+ 2$^1$+2$^1$

=5 $^2$(1)+ 2 +2 $^1$

$\boxed{= 5\,^2 + 2 + 2}$ ---------②

$\frac{^2}{} = \frac{}{} \left(\frac{}{}\right) = \frac{}{} (6\,^2 + 10xy + \,^2)$

$^2$

$^2_2$=2 × 6 × $^{2-1}$ +1×10× $^{1-1}$y +0

$^2_2$=12 +10 y

$^2_2$=12x +10(1)y

$\boxed{\frac{^2}{_2} = 12x + 10y}$ -----------③

$\frac{^2}{_2} = \frac{}{} \left(\frac{}{}\right) = \frac{}{} (5\;^2 + x2y + 2y)$

$\frac{\partial^2 z}{\partial x \partial y} = 0 + x1\times 2 \times y^{1-1} + 1 \times 2 \times y^{1-1}$

$\frac{\partial^2 z}{\partial x \partial y} = x2y^0 + 2y^0$

$\frac{\partial^2 z}{\partial x \partial y} = x2(1) + 2(1)$

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = x2 + 2}$$ ------④

$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(5x^2 + x2y + 2y)$

$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(5x^2 + x2y + 2y)$

$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(5x^2 + x2y + 2y)$

$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(5x^2 + x2y + 2y)$

$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial x}(2 \times 5 \times x^{2-1} + 1 \times x^{1-1}2y + 0)$

$\frac{\partial^2 z}{\partial y \partial x} = (10x^1 + 1x^0 2y)$

$\frac{\partial^2 z}{\partial y \partial x} = (10x + 1(1)2y)$

$\frac{\partial^2 z}{\partial y \partial x} = (10x + 2y)$

$$\boxed{\frac{\partial^2 z}{\partial y \partial x} = 10x + 2y}$$ -------⑤

$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(\;2\;\;2\;)$

$\frac{\partial^2 z}{\partial y^2} = 0 + 10\,1\times x^{1-1} + 1\times 2 \times y^{2-1}$

$\frac{\partial^2 z}{\partial y^2} = 10x^0 + 2y^1$

$\frac{\partial^2 z}{\partial y^2} = 10x^0 + 2y^1$

$\frac{\partial^2 z}{\partial y^2} = 10(1) + 2y$

2

---------⑥

4. b. If $z = 2^3 + 5^2 y + x^2 +$ $^2$ verify that

=

Soln:

$=3\times2\times$ $3-1+2\times5\times$ $2-1$ $+1\times$ $1-1$ $2+0$

$=6^2+10^1$ $+1^{02}$

$=6^2+10$ $+1(1)^2$

$= 6^2$ $+10$ $+1^2$ ----------①

$=2^3+5^2+$ $^2+^2$

$=0+5^2 1\times$ $1-1+2\times$ $2-1+2\times$ $2-1$

$=5^{20}+$ $2^1+2^1$

$=5^2(1)+$ $2+2$

$=5^2(1)+$ $2+2$

$= 5^2$ $+2+2$ ----------②

$(6^2 + 10xy + ^2)$

$= 2\times6\times$ $2-1$ $+1\times10\times$ $1-1$ $y+0$

$= 12$ $+10(1)y$

2

$_2 = 12\ ^1 + 10y$

2

= ( )

(5 $^2$ + x2y + 2y)

$^2_2$ = (0 + x1× 2 × 1−1 + 1 × 2 × 1−1)

$^2$ = x2 0 + 2 0

$^2_2$ = x2(1) + 2(1)

$^2_2$ = x2 + 2

---------------③

$^2$ = ()

= (5 $^2$ + x2y + 2y)

------------④

$^2$ =2×5× $^{2-1}$+1× $^{1-1}$2 +0

$^2$ =10 $^1$+1 $^0$2

$^2$ = 10x+1(1)2

$^2$ = 10x+2

-----------(5)

$^2$ = ()

= (6 $^2$ + 10xy +1 $^2$)

$^2$ = 0 + 10 x 1× 1−1 + 2 × 1 × 2−1

$^2$ = 0 + 10 x 1× 1−1 + 2 × 1 × 2−1

$\frac{2}{} = 10\,x^{0} + 2^{1}$

$\frac{2}{} = 10\,x(1) + 2$

$\boxed{\frac{2}{} = 10\,x + 2}$ ------------⑥

**Part – C**

y= $^{3}$- 2 $^{2}$+ x + 4

= $^{3}$−2 $^{2}$+ +4

=3 $^{2}$−4 +1+0

=3 $^{2}$−4 +1

x - $\frac{1}{3}$ x - 1

$\boxed{x = \frac{1}{3} \quad x = 1}$

$\frac{2}{}$= 3.6x - 4

$\frac{2}{}$=6($\frac{1}{3}$)-4

$\frac{2}{}$=2-4

=-2<0

Function of x = $\frac{1}{3}$ have maximum value

$\frac{2}{}$= 6x - 4

$\frac{2}{}$= 6(1) -4

$\frac{2}{}$=6-4

$\frac{}{}$= 2

2

Function of x = 1 have minimum value

y = $x^3 - 2x^2 + x + 4$

y = $\left(\frac{1}{3}\right)^3 - 2\left(\frac{1}{3}\right)^2 + \frac{1}{3} + 4$

y = $\frac{1}{27} - 2\times\frac{1}{9} + \frac{1}{3} + \frac{4}{1}$

y = $\frac{1}{27} - \frac{2}{9} + \frac{1}{3} + \frac{4}{1}$

$\frac{1-6+9+108}{27}$

= $\frac{112}{27}$

Maximum value = $\frac{112}{27}$

x = 1

y = $x^3 - 2x^2 + x + 4$

y = $(1)^3 - 2(1)^2 + 1 + 4$ y = 1 - 2 + 1 + 4

y = 4

Minimum value = 4

# UNIT – III
# INTEGRAL CALCULUS

**Integration – meaning:**

i) The inverse (reverse) process differentiation is known as "Integration"

ii) It is denoted by "∫" there are two types of integration. Namely

1. Indefinite Integral  2. Definite Integral

**Difference between differentiation and Integration:**

i) In differentiation, $= n^{-1}$

ii) In integration, $= {}^{+1} + c$

$+1$

**Integration and its Rules:**

**Rules:**

When we find out the Integration we have to remove the integration symbol ∫ and put the constant sign "C" lastly.

**Rules I:**

Integration of dx

$\int \quad = \quad +$

**Rule II:**

Integration of k.dx

$\int . \quad = \quad +$

**For example:**

Integration of 3 dx

$\int 3. \quad =3\int \quad =3 +$

**Rule III:**

Integration of

$\int \quad . \quad = \quad {}^{-1}{}_{+1} +$

For example:

1. $\int {}^{3} .$

$= \frac{3+1}{3+1} + c$

$= \frac{4}{4} + c$

2. Evaluate $\int \frac{1}{5}$ Soln:

1

$= \frac{-5+1}{-5+1} + c$

$= \frac{-4}{-4} + c$ $-4$

$= \frac{1}{-4} -4 +$

Ans: $\int \frac{1}{5} = -\frac{1}{4} -4 +$

1

3. $\int \sqrt{} \, . dx = \int 2. dx$

1

$= \frac{}{1} + 1 + c$ 2

3

$= 3 + c$ 2

$\int \sqrt{} \, . dx = 3^2 + c$ 3 2

Ans:

4. Evaluate $\int -7$

$= \frac{-7+1}{-7+1} + c$

$= \frac{-6}{} + c$ $-6$

$= \frac{1}{-6}$ $-6$ $+ c$

$\int -7 = \frac{1}{-6} -6 + c$

Ans:

**Rule IV:**

Integration of sum (Addition & subtraction)

**Example:1**

Evaluate $\int ( ^2 - x + 1)\, dx$

$= \int ^2 . \quad - \int . \quad + \int 1.$

$= ^{2+1} \frac{}{2+1} - ^{1+1} \frac{}{1+1} + +$

$\int (2 - x + 1)\, dx = ^3 - ^2 + +$

$_3 \quad _2$

Ans:

1

2. Evaluate $\int ( _3 + _2 - + - 1)$

$\int ( 3 +$

$_2 - -1)$

$\int ( ^3 + \ ^{-2} \quad - 1)$

$\int ^3 . \quad + \int ^{-2} . \quad - \int \quad . \quad - \int 1.$

$3+1 _{3+1} + \ -2+1 _{-2+1} - \ 1+1 _{1+1} - \ +$

Ans:

$\int ( ^3 + \quad \frac{1}{2} - - 1) = \quad \frac{^4}{4} + \frac{^{-1}}{-1} - \frac{^2}{2} - +$

**Rule V:**

Integration of multiplication

**Example: 1**

$\int 2 .$

Soln:

$2 \ \int .$

$=2\frac{1+1}{} +$

$= \frac{2^2}{} + = 2 +$

Ans: $\boxed{\int 2. = ^2 \quad +}$

2. Evaluate $\int 4\ ^6$.

Soln:

$4 \int {}^6.$

$= 4\frac{^{6+1}}{6+1} + c$

$= \frac{4^7}{7} + c$

Ans: $\boxed{4 \int {}^6. = \frac{4}{7} + c}$

3. Evaluate: $\int 3\ ^2$.

Soln:

$3 \int {}^2.$

$= 3\frac{^{2+1}}{} + c$

$= \ ^3 +$

Ans:

$\int 3^2. = \ ^3 +$

**Rule VI:**

Integration of substitution (product)

**Example: 1**

$\int\ 2\ ^2 (\ ^3 + 5)^3 dx$

Soln:

u = $^3 + 5$

= 3 $^2$

=

$3^2$.

$= $

/ dx =

3 2

Substituting in $\int 2\,^2\,(^3 + 5)^3$dx

We have

$\int 2\,^2\,.\,^3$.dx

Putting dx = 3 2

We have

$\int$ 22.3.

3 $^2$

2

$\int$3. 3.

$= \frac{2}{3} \cdot \frac{^{3+1}}{3+1} +$

$= \frac{2}{3} \cdot \frac{^4}{4} +$

$= \frac{1}{3} \cdot \frac{^4}{2} +$

$= \frac{^4}{6} +$

Putting u = $^3 + 5$

$\frac{(^2+5)^4}{6}$ + c

Ans: $\int 2\,^2\,(^3 + 5)^3$dx = $\frac{(\;^2+5)^4}{6}$ + c

2. Evaluate $\int 4\,^2\,(^3 + 5)^3$dx

Soln:

$= \quad . \quad ^{-1}$

$u = {}^{3} + 5$

$/ \_\_ = 3^2$

$3^2.$ $\quad = $

$/$

$= 32$

Substituting in $\int 4\,{}^{2}\,({}^{3} + 5)^3 dx$

We have :

$\int 4^2\,{}^3.$

Putting

$= \quad 3\,{}^2$

We have

$\int \quad 423.$ $\quad 3\,{}^2$

$= \int 4.\,3.$

$= 4 \,.\, {}^{3+1} + c$

$= \quad 4\,.\,{}^4 + c$

$= \quad 1\,.\,{}^4 + c$

$= \quad 4 + c$

Putting ~~u =~~ $^3 + 5$

~~$({}^3+5)^4 + c$~~

Ans: $\int 4\,2\,(3 + 5)3 dx = (3+5)4 + c$

**Rule VII:**

Integration of quotient

Example : 1

2

u = ( 2 − 2)

−2

2x.dx = du

= 2

Substituting ∫ ( 2 2 −2)2 .

= ∫ 2 2.

= ∫ 2 . 1 2.

= ∫ 2 . −2.

Putting dx = 2

= ∫ 2 . −2.

2

= ∫ −2.

= −2+1 + c

−2+1

= −1 + c −1

Substitute u = ( 2 − 2)

= ( 2−2) −1 + c

−1

2

Ans: ∫ ( 2−2)2 . = ( 2 − 2)−1 +

1. Evaluate ∫ ( 2 8−5)4 . Soln:

U=( 2−5)

$= 2$

$2x.dx = du$

$= 2$

Substituting

$8$

$(\ -5)$

$= \int 8_4 .$

$= \int 8_4 .$

$= \int 8 . \ 4.$

$= \int 8 . ^{-4}.$

Putting dx = $2$

$= \int 8 . ^{-4}.$

$2$

$= \int 4. ^{-4}.$

$= 4. \frac{^{-4+1}}{-4+1} + c$

$= 4. \frac{^{-3}}{-3} + c$

Substitute u = $(\ ^2 - 5)$

$= \frac{4(\ ^2 - 5)^{-3}}{-3} + c$

Ans: $\int \frac{8}{(\ -5)} . = \frac{4(\ ^2 - 5)^{-3}}{-3} + c$

**Rule VIII:**

**Power Function**

**1**. y =

In integration

Y = ∫ .dx Y = +
c

**2.** Evaluate

∫ +2.dx Y = +2
+ c

**3**.Solve∫ ( + 1 ).dx

Soln: ∫( + −2)
∫ . +∫−2.

= + −2+1 + c
−2+1

= + −1 + c
−1

Ans: ∫( + 1 ).dx = + −1 + c

**Rule:IX**

**Integration of log function**

**Example: 1**

Solve: ∫ . dx

Ans: logx +c

**Example:2**

Evaluate:∫ 4 .dx

Soln:

4 ∫ 1 . dx

Ans:

4logx +2

**3.** Evaluate ∫ +2$^2$ .dx

u = x+2

= 1

du = 1.dx

dx =

substituting in ∫ +2 .dx

2logu+c

Ans: 2log (x+2)+c

**II. Definite Integration:**

i) The area between two curves (a&b) and the x-axis we have definite solution ie: a number (n) or a value of the constant C.

ii) It came to be called "The Definite Integral of f(x) from x = a to x = b. It is denoted by

iii)

Example:1

Evaluate

Soln:

a = 1 b = 2

$=[\frac{8}{3}-\frac{1}{3}]$

$=\frac{7}{3}$

3

Soln:

$(3)^3$

$= [\frac{3(9)}{2} - \frac{3(4)}{2}]$

$= [\frac{27}{2} - \frac{12}{2}]$

$= [\frac{15}{2}]$

Ans: $\int_2^3 3\,. = [\frac{15}{2}]$

1. Evaluate

$\int (x^3 + 5x + 3)$

Soln:

a = 3 b = 4

$[\frac{x^{3+1}}{3+1} + \frac{5x^{1+1}}{1+1} + 3x]_3^4$

$[\frac{x^4}{4} + \frac{5x^{1+1}}{1+1} + 3x]_3^4$

$[\frac{x^4}{4} + \frac{5x^2}{2} + 3x]_3^4$

$= [\frac{4^4}{4} + \frac{5\times4^2}{2} + 3\times4] - [\frac{3^4}{4} + \frac{5\times3^2}{2} + 3\times4]$

$= [\frac{256}{4} + 80 + 12] - [81 - 45 + 9]$

= 4,2,1 LCM

$\frac{256}{4} \times \frac{1}{1} = \frac{256}{4}$

80 2 160

2 ×2 = 4

12 4 48

1 ×4 = 4

256+160+48 464

[ 4 ] = 4

81 1 81

4 ×1 = 4

45 2 90

2 ×2 = 4

9 4 36

1 × 4 = 4

81 − 90+36

4

= 27

464 27

= 4 − 4

= 464−27

4

= 437

4

Ans: ∫ 4 ( 3 + 5 + 3) = 437

3 4

2. ∫ 2 3.dx

1

Soln:

= [ +1

+1

∫ 2 3.dx

1

( ) 2

3+1

3 + 1 1

a = 1 , b = 2

+1 +1

+1 +1

$2^{3+1}$ $1^{3+1}$

3+1 - 3+1

4 4

16 - 1

$= \frac{4}{15} 4$

Ans: $\int^{2}$ 3.dx = 15

1 4

## Cost and Revenue Function

Tc = ∫ .

### Example: 1

If Mc = 4 – 2x - 2 find Tc

Given:

Mc = 4 – 2x - 2

Find:

Tc

Soln:

Tc = ∫ .

Tc = ∫(4 − 2 − 2)

Tc = 4x - 2 1+1 - 2+1

Tc = 4x - 2 2 - 3 + c 1+1 2 2+1 3

Tc = 4x - 2 - 3 + 3

### Example: 2

If Tfc = 40, Mc = 2 + 4x + 2 2 find Tc

Given:

Tfc = 40, Mc = 2 + 4x + 2 2

Find:

Tc

Soln:

Tc = ∫ .

Tc = ∫( 2 + 4x + 2 $^{2}$)

Tc = 2x + 4 $^{1+1}$ - 2 $^{2+1}$ + c

1+1 2+1

Tc = 2x + 4 $^{2}$ - 2 $^{3}$ + c

2 3

Ans: Tc = 2x + 2 $^{2}$ + 2 $^{3}$ + 40

3

**Example : 3**

If Tfc = 40, mc = 4+ 4x – 8x - 4 $^{2}$ find Tc, Ac &Avc

Given:

Tfc = 40, mc = 4+ 4x – 8x - 4 $^{2}$

Find :

Tc, Ac &Avc

Soln:

Tc = ∫

Ac =

Avc =

Tc = 4x + $\frac{4^{1+1}}{1+1}$ – $\frac{8^{1+1}}{1+1}$ – $\frac{4^{2+1}}{1+1}$ +

4 $^{2}$ 8 $^{2}$ - 4 $^{3}$

Tc = 4x + 2 – 2 3 +

$\boxed{\text{Tc} = 4x + 2^{2} - 4^{2} - \frac{4^{3}}{3} + 40}$

4x+2–4–4 3 +40

Ac =

3

Ac = 4 + 2x – 4x - 4 2 + 40

3

Avc = (Tfc not include)

Tvc = 4x - 2 2 – 4 3

3

4x–2 2–4 3

Avc =

3

Avc = 4 – 2x - 4 3 2

4 2

3

Avc = 4 – 2x -

**Ans:**

Tc = 4x + 2 2 - 4 2 - + 40

Ac = 4 + 2x – 4x - + 

Avc = 4 – 2x - 

4+ 4x – 8x - 4 2

Tc = ∫

= 4x + 4 2 - 8 2 - 4 3 2 2 3

= 4x + 2 2 – 4 2 – 4 3

Tc = 4x - 2 2 – 4 3 + 40

Ac =

Ac = 4 – 2x– 4 3 + 40

3

4x – 2 – 4 3

Avc =

**Revenue Function**

**Example: 1**

If MR = 50 – 2Q find TR

TR=∫

= 50Q – 2 1+1 +c

2 2 1+1

= 50Q - 2 +c

TR=50Q–Q+C

**Example : 2**

If MR = 6 - 3 2 find TR and AR

Given:

MR=6-3 2

Find:

TR=?

AR= ?

Soln:

TR=∫ .

AR =

TR=∫6−3 $^2$.

= 6x - 2+1 + c

3 3

= 6x - 3 + c

TR = 6x - 3 + c

AR =

AR = 6x − 3+

AR = 6 - 2 + c

Ans:

TR = 6x - 3 + c

AR = 6 - 2 + c

3. If MR = 3 $^2$ − 2 + 4 find TR and Demand Function Given:

MR=3 $^2$−2 +4

Find :

TR and Demand function

Soln:

TR=∫ ( ).

(AR) Demand function Q =

TR=∫(3 $^2$−2 +4)

= 3 3 − 2 2 +4x + c

3 3

TR = 3 2 + 4 + c

TR = 3− 2+4 + c

Q = 3− 2+4

Q= 3− +4

Ans:

TR = $^3 - \ ^2 + 4 \ + c$

Q= $^2 -$ $+4$

**<u>Consumer Surplus and Producers Surplus</u>**

**<u>(Efh;Nthh; vr;rk; cw;gj;jpahsh; vr;rk;</u>**

**<u>A. Consumer Surplus:</u>**

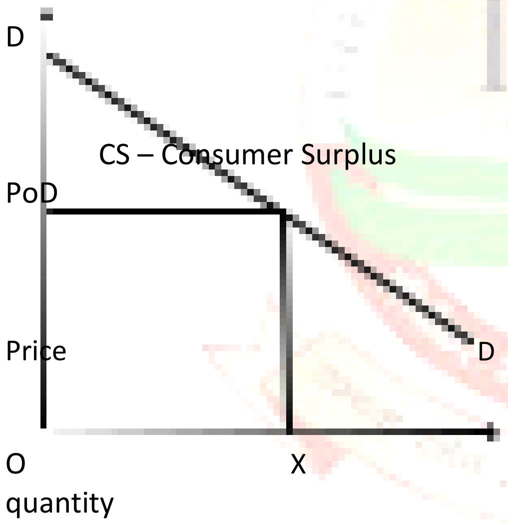
i) “Consumer’s Surplus is the difference between Potential price and actual price” said

ii) Demand function: (Njitr;rhh;G) Tassing

a) Demand function for a good refers to the amount of that good that will be bought by people in a given price.

b) That is p = f(x) is the demand function for a commodity.

**Diagram:**




**Diagramatic Explanation:**

i) We assume that a consumer purchased xo quantity of good at po price iepo = f(xo)

ii) Hence the total expenditure of the consumer = poxo

iii) In integration it is denoted by $\int$ (or) $\int$ ( ).

iv) we have consumer surplus = $\int$ . – po xo

1. If the demand function is p = 35 – 5x - 5 $^2$ and the demand xo is 2 find the consumer surplus. Given:

P = 35 – 5x -5 $^2$ Find:

Demand xo is

2 Soln:

P = 35 – 5x - 5 $^2$

Put x = 0 in demand function.

$Po = 35 - 5(2) - 5(2)^2$

$= 35 - 10 - 5(4)$

$= 35 - 10 - 20$

$= 5$

Po = 5

Po = 5 &xo = 2

consumer surplus = $\int$ . $-$ po

$\int_0^2 35 - 5 - 5^2$. $= 5^2$

$\left[35 - \frac{5^{1+1}}{1+1} + \frac{5^{2+1}}{2+1}\right]_0^2$ -10

$\left[5 - \frac{5^2}{2} - \frac{5^3}{3}\right]_0^2$ -10

$= \left[35(2) - \frac{5(2)^2}{2} - \frac{5(2)^3}{3}\right]$ -10

$= \left[70 - \frac{5\times4}{2} - \frac{5\times8}{3}\right]$ -10

$= \left[70 - \frac{20}{2} - \frac{40}{3}\right]$ -10

$= \left[\frac{420-60-80}{6}\right]$ -10

$= \frac{280}{6} - \frac{10}{1}$

$= \frac{280-60}{6}$

= 36.67

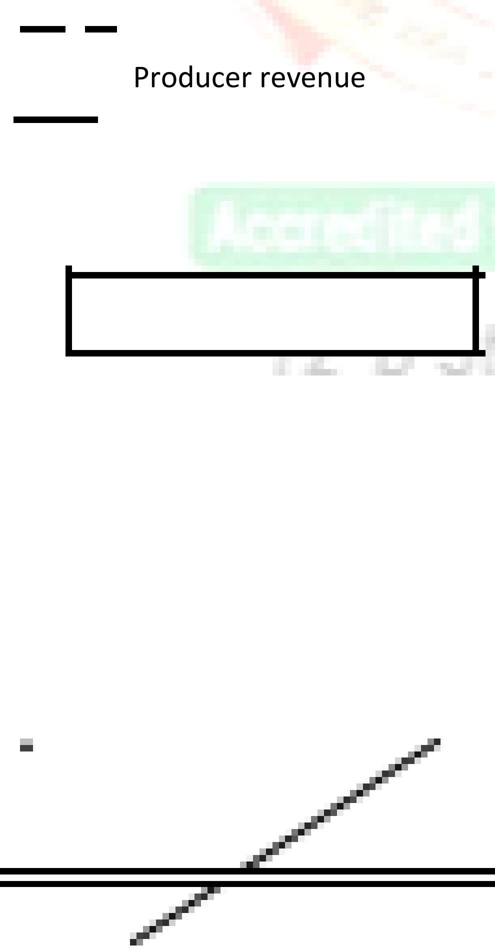
Ans: Consumer Surplus = 36.67

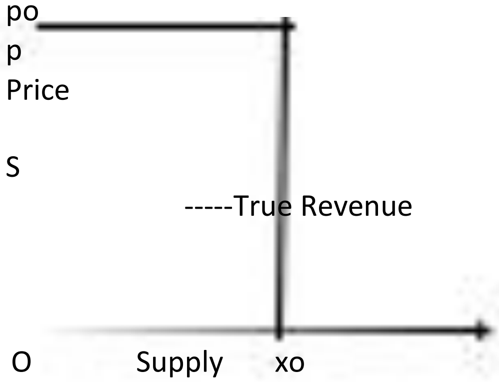
**Producer's Surplus:**

i) Producer's Surplus is the difference between producer is willing to sell the quantity of goods (ieproducers revenue = poxo) at a given price and line true revenue is selling the quantity of goods (ie f(x) dx)

ii) Producers surplus = poxo - $\int$ ( ) .

Diagram:






1. The supply function for a commodity p = $2\ ^2 - \ + 4$ when the price is 10, the producers

surplus.

Soln:

P=2 $^2 - \ +4$

Po = 10

10=2 $^2 - \ +4$

2 $^2 - \ +4=10$

2 $^2 - \ =10-4$

$2^2 - \ =6$

2 $^2 - \ -6=0$

(x – 2) = 0 (2x+3) = 0

x = 2, x = $\frac{-3}{2}$

$\boxed{x = 0}$

ps = (10) (2) - $\int_0^2 (2\ ^2 - \ + 4)$

ps = 20 - $\int_0^2 (2\ ^2 - \ + 4)$

ps = 20 - [ $\frac{2\ ^{2+1}}{2+1} - \frac{\ ^{1+1}}{1+1} + 4\ ]_0^2$

ps = 20 -[ $\frac{2\ ^3}{3} - \frac{\ ^2}{2} + 4\ ]_0^2$

ps = 20 -[ $\frac{2(2)^3}{3} - \frac{(2)^2}{2} + 4(2)$ ]

ps = 20 -[ $\frac{2(8)}{3} - \frac{4}{2} + 8$ ]

ps = 20 -[ $\frac{16}{3} - \frac{4}{2} + 8$ ]

3,2 con is $3\times 2 = 6$

$\frac{16}{3} \times \frac{4}{2} = \frac{32}{6}$

$\frac{4}{2} \times \frac{3}{3} = \frac{12}{6}$

$\frac{8}{1} \times \frac{6}{6} = \frac{48}{6}$

$\frac{32-12+48}{6}$

$\frac{68}{6}$

=

=

ps = $\frac{20}{1} - \frac{68}{6}$

1,6 Lcm is = 6

$\frac{20}{1} \times \frac{6}{6} = \frac{120}{6}$

$\frac{68}{6} \times \frac{1}{1} = \frac{68}{6}$

$\frac{120-68}{6} = \frac{52}{6}$

$-8.67$

## UNIT – III

**Part - A**

1. $\int$ - dx ia value of

a) +c b)1+c c) x +c d)

2. The value of $\int$ 5 dx is

a) 5x+c b) 5x c) dx d) 5+c

3. value of $\int$ .

a) x b) — c) x+c d)None of these

4. value of $\int 6\ ^{5}$.

a) +c b) c) $\frac{}{6}$+c d) 30 4

5. $\int$ .

+1

a) b) c) d) 1x

6. If ∫ =

a) x+c b) x c) 0 d) - c

7. If – dx =

a) log x b) logx+c c) logx – c d) 0

8. Consumer Surplus =

0

a) $\int_0$ ( ) – b) $\int_0$ ( ) c) $\int_0$ . d) $\int_0$ ( ) –

## Part – B

1. Evaluate $\int(4\ ^3 + 3\ ^2 - 2)$
2. Evaluate ∫2 (3 – 2)?dx
3. If the MR function MR = 100 – 4Q find the total revenue function.
4. Evaluate ∫ 3 ² –
5. Find ∫ [ + ¹ ] dx
6. Compute TC for the MC function MC = 2+6x - 4 ², if total fixed cost is 50.
7. If mc = 3 -2x - ² find the TC.
8. Find $\int 3\ ^2 + 9 + 10$
9. Find the value of $\int^0( - 19)$
10. Evaluate $\int(^3 + ^{-1} - - 1)$
11. Evaluate $\int_1 ^2(\ ^3 - 2$ – 3)

## Part – C

1. Evaluate $\int_{-1}^{1}(2\ ^2 - \ ^3)$
2. The supply function for a commodity p = 2 + ² find the producer's surplus when price is Rs.18
3. Given the total cost function c = 15 ²+ 10x +60 find the average cost and marginal cost function.
4. Evaluate $\int 4\ ^2 (\ ^3 + 5)^3$
5. Explain producers surplus with the help of integration.
6. The demand functions for a commodity p = 30 – 2 the supply function p = 3D find the consumer surplus.
7. The supply functions for a commodity p = 2 + ² find producers surplus when price is Rs.18.
8. Find $\int 5\ ^2 + 2 - 10$
9. Find the value of $\int_2 ^2( - 10)$
10. find $\int(2 - 5)(36 + 4)$
11. If the demand function is p = 8 – 2x and the supply function is p = 2 + x, what will be the consumer surplus?
12. Evaluate ∫ dx
13. The demand function p = 30 – 2x the supply function 2p = 5 +x find consumer surplus?

# UNIT – IV
# MATRIX (mzpfs;)

**Meaning:**

The matrix is a square or a rectangular array of members, usually represented by enclosing the array by. The brackets. The brackets may be ( ) or [ ]. The number of rows (m) and columns (n) are called Dimensions or rank or order of a matrix.

In short, matrix is a collection of vectors this mathematical method was indices by the English mathematician Aurthorcayley 1858. The numbers in a are also called its elements. These elements may be real or complex are elements.

Basic Concepts:

1. Denotions:

Usually, matrixes are denoted by capital letters viz. A,B,C and so on X,Y,Z and the elements on denoted by small letters a, b, c and so on x, y, z order of matrix. A → ×n = 2× 2 Constraction of a matrix

Eg:

11 12 13……… 1

21 22 23……… 2 1

I 31 32 33……… 3 I

I ………………………………………… I

I I

I… … … … … … … … … … … … …… Imxn= [aij] here matrix A = aij

I ………………………………………… I

I ………………………………………… I

I ………………………………………… I

[ 1 2 …………]

are I rows and $j^{th}$columns.

**2. Rows and Columns:**

The elements of the rows GO from left to Right horizontally and the elements of a column Go from top to pattam vertically

Hence the number in horizontal lines are called the rows ($a_{11}$ $a_{12}$ 0 ....... $a_{1n}$) and the

11

12 1

I I

31

I I

numbers in vertical line are I . I

I . I

I . I

I . I

[ 1]

**3. Diagonal Elements:**

The elements (a11, a22, a33………amn) are called as primary diagonal elements.

Elements above the diagonal elements (a12, a13 ………a1n, a23 ……….a2n and ........ a3n are called upper diagonal elements.

Elements lower diagonal elements a21, a32 ……….amn, a31 ........... am3 and am1, am2 are called lower diagonal elements.

**4. Dimension or Rankor order of a matrix:**

The number of rows (m) and columns (n) in a matrix is called the dimension or rank or order of a matrix. A matrix contains (m) rows and (n) columns. Is called mxn. Order or rank of a matrix.

Ex: 1

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2x2}$$

There are two rows and two columns. Hence mxn = 2x2

Ex: 2

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3x2}$$

There are three rows and two columns. Hence mxn = 3x2

**Types of matrix: (mzpapd; tiffs;)**

**1. Single matrix: (xUcWg;Gmzp)**

Matrix contain a single elements. It is called a single matrix.

Ex: 1

[1]

A = $[1]_{1x1}$

There is only one elements. Hence

mxn = 1x1

**2. Row matrix:**

A matrix with a single row is called a row matrix.

Ex:

A = $[3\ -5\ 7]_{1x3}$

**3. Column matrix:**

A matrix with a single column is called a column matrix.

Ex:

$$A = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}_{3x1}$$

**4. Zero or nullor empty matrix**:

If every elements of a matrix is zero. It is called a null or zero matrix. It is denoted by zero.

Ex:

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2x2}$$

**5. Square matrix:**

A matrix with equal number of rows and columns is called a square matrix. (ie) m = n

Ex:

$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}_{2x2}$

$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3x3}$

**6. Diagonal matrix:**

A square matrix in which all the upper and lower diagonal elements are zero except on the (edding is called diagonal matrix)

Ex:

$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3x3}$

**7. Scale are matrix:**

A diagonal matrix in which all the diagonal elements are equal is called a scale are matrix.

Ex:

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2x2}$

$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3x3}$

**8. Unit or identity matrix:**

A square matrix in which all the diagonal elements are one is called the and unit or identity matrix. It is denoted by identity matrix. It is denoted by identify matrix I.

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2x2}$

$I = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3x3}$

**9. Equal matrix:**

Two matrixes are equal (A=B), if they have the same order and that their corresponding elements are equal.

Ie. aij = bij

That is called equal matrix.

Ex:

$A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

$B = \begin{bmatrix} 2 & 4 \\ & 8 \end{bmatrix}_{2x2}$

### 10. Rectangular matrix:

A matrix in which number of rows is greater then is number of columns or the number, of rows is less then the number of columns. It is called rectangular matrix. m> (or) m<

Ex:

B=[1 23

4 32 ]2x3

1 4

A=[2 3 ]3x2

3 2

### 11. Symmetric matrix:(rkr;rPh; mzp)

The square in which all the lower and upper diagonal elements are the same except on the leading diagonal is called symmetric matrix.

Ex:

A = [ ]3x3

5 43

B=[ 4 62 ]3x3

3 2 7

Here aij = +aij or= + A

### 12. Skew symmetric matrix:

A square matrix in which all the upper diagonal elements are positive and the lower diagonal elements are negative except on the leading diagonal is called a skew symmetric matrix.

Ex:

1. A = [ 0 −3 3 0 ]

0 7 9

2.A=[−7 0 3]3x3

−9 −3 0

Here +aij = - aij or = -A

### 13. Item potend matrix:

A symmetric matrix for which = A and 2 = A is called item potend matrix.

Ex:1

A=[1 0]

0 1

= [ 1 0]

0 1

Hence = A

**14. Singular matrix: (xUikad; mzp)**

A square matrix in which the determinand is equal to zero (ie) [A] = 0 it is called a singular matrix.

Ex:

$$A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}_{2\times 2}$$

[A]

**Matrix operations**

Equality

Two matrices are equal if and only if

- The order of the matrices are the same
- The corresponding elements of the matrices are the same

**Addition**

- Order of the matrices must be the same
- Add corresponding elements together
- Matrix addition is commutative
- Matrix addition is associative

**Subtraction**

- The order of the matrices must be the same
- Subtract corresponding elements
- Matrix subtraction is not commutative (neither is subtraction of real numbers)
- Matrix subtraction is not associative (neither is subtraction of real numbers)

**Scalar Multiplication**

**A scalar is a number, not a matrix.**

- The matrix can be any order
- Multiply all elements in the matrix by the scalar
- Scalar multiplication is commutative
- Scalar multiplication is associative

**Zero Matrix**

- Matrix of any order
- Consists of all zeros
- Denoted by capital O
- Additive Identity for matrices
- Any matrix plus the zero matrix is the original matrix

**Matrix Multiplication**

$A_{m\times n} \times B_{n\times p} = C_{m\times p}$

- The number of columns in the first matrix must be equal to the number of rows in the second matrix. That is, the inner dimensions must be the same.
- The order of the product is the number of rows in the first matrix by the number of columns in the second matrix. That is, the dimensions of the product are the outer dimensions.
- Since the number of columns in the first matrix is equal to the number of rows in the second matrix, you can pair up entries.

- Each element in row *i* from the first matrix is paired up with an element in column *j* from the second matrix.
- The element in row *i*, column *j*, of the product is formed by multiplying these paired elements and summing them.
- Each element in the product is the sum of the products of the elements from rows *i* of the first matrix and column *j* of the second matrix.
- There will be *n* products which are summed for each element in the product.

See a complete example of matrix multiplication.

Matrix multiplication is not commutative

- Multiplication of real numbers is.
- The inner dimensions may not agree if the order of the matrices is changed.
- Do not simply multiply corresponding elements together
- Since the order (dimensions) of the matrices don't have to be the same, there may not be corresponding elements to multiply together.
- Multiply the rows of the first by the columns of the second and add.
- There is no matrix division
- There is no defined process for dividing a matrix by another matrix.
- A matrix may be divided by a scalar.
- Identity Matrix
- Square matrix
- Ones on the main diagonal
- Zeros everywhere else
- Denoted by I. If a subscript is included, it is the order of the identity matrix.
- I is the multiplicative identity for matrices
- Any matrix times the identity matrix is the original matrix.
- Multiplication by the identity matrix is commutative, although the order of the identity may change

*Identity matrix of size 2*

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

*Identity matrix of size 3*

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

*Properties of Matrices*

| *Property* | *Example* |
|---|---|
| *Commutativity of Addition* | *A+B=B+A* |
| *Associativity of Addition* | *A+(B+C)=(A+B)+C* |
| *Associativity of Scalar Multiplication* | *(cd) A = c (dA)* |
| *Scalar Identity* | *1A=A(1)=A* |
| *Distributive* | *c (A + B) = cA + cB* |
| *Distributive* | *(c + d) A = cA + dA* |
| *Additive Identity* | *A+O=O+A=A* |
| *Associativity of Multiplication* | *A (BC) = (AB) C* |
| *Left Distributive* | *A(B+C)=AB+AC* |
| *Right Distributive* | *(A+B)C=AC+BC* |
| *Scalar Associativity / Commutativity* | *c (AB) = (cA) B = A (cB) = (AB) c* |
| *Multiplicative Identity* | *IA=AI=A* |

*Properties of Real Numbers that aren't Properties of Matrices*

*Commutativity of Multiplication*

- *You can not change the order of a multiplication problem and expect to get the same product. AB≠BA*
- *You must be careful when factoring common factors to make sure they are on the same side. AX+BX = (A+B)X and XA + XB = X(A+B) but AX + XB doesn't factor.*

*Zero Product Property*

- *Just because a product of two matrices is the zero matrix does not mean that one of them was the zero matrix.*

*Multiplicative Property of Equality*

- *If A=B, then AC = BC. This property is still true, but the converse is not necessarily true. Just because AC = BC does not mean that A = B.*

- *Because matrix multiplication is not commutative, you must be careful to either pre-multiply or post-multiply on both sides of the equation. That is, if A=B, then AC = BC or CA = CB, but AC≠CB.*

*There is no matrix division*

- *You must multiply by the inverse of the*

*matrix Evaluating a Function using a Matrix*

*Consider the function $f(x) = x^2 - 4x + 3$ and the matrix A*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

*The initial attempt to evaluate the f(A) would be to replace every x with an A to get $f(A) = A^2 - 4A + 3$. There is one slight problem, however. The constant 3 is not a matrix, and you can't add matrices and scalars together. So, we multiply the constant by the Identity matrix.*

$$f(A) = A^2 - 4A + 3I.$$

*Evaluate each term in the function and then add them together.*

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$-4A=-4 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -4 \\ -12 \end{bmatrix}$$

$$3I=3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$f(A) = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} + \begin{bmatrix} -4 \\ -12 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 3 & 9 \end{bmatrix}$$

*Factoring Expressions*

*Some examples of factoring are shown. Simplify and solve like normal, but remember that matrix multiplication is not commutative and there is no matrix division.*

*2X+3X=5X*
*AX + BX = (A+B)X*
*XA + XB = X(A+B)*
*AX + 5X = (A+5I)X*
*AX+XB does not factor*

*Solving Equations*

*A system of linear equations can be written as AX=B where A is the coefficient matrix, X is a column vector containing the variables, and B is the right hand side. We'll learn* how to solve this equation in the next section.

If there are more than one system of linear equations with the same coefficient matrix, then you can expand the B matrix to have more than one column. Put each right hand side into its own column.

Matrix Multiplication

Matrix multiplication involves summing a product. It is appropriate where you need to multiply things together and then add. As an example, multiplying the number of units by the per unit cost will give the total cost.

The units on the product are found by performing unit analysis on the matrices. The labels for the product are the labels of the rows of the first matrix and the labels of the columns of the second matrix.

**Matrix Operations**

**Determinant of a Matrix**

The determinant of a matrix is a special number that can be calculated from a square matrix.

A Matrix is an array of numbers:

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix}$$

A Matrix
(This one has 2 Rows and 2 Columns)

The determinant of that matrix is (calculations are explained later):

$3\times6-8\times4=18-32=-14$

What is it for?

The determinant helps us find the inverse of a matrix, tells us things about the matrix that are useful in systems of linear equations, calculus and more.

**Symbol**

The symbol for determinant is two vertical lines either side.

Example:

$|A|$ means the determinant of the matrix A

(Exactly the same symbol as absolute value.)

**Calculating the Determinant**

First of all the matrix must be square (i.e. have the same number of rows as columns). Then it is just basic arithmetic. Here is how:

For a 2×2 Matrix

For a 2×2 matrix (2 rows and 2 columns):

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is:

$$|A| = ad - bc$$

*"The determinant of A equals a times d minus b times c"*

It is easy to remember when you think of a cross:

- Blue is positive (+ad),
- Red is negative (−bc)

Example:

$$B = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$$

|B|= 4×8 − 6×3
= 32−18
= 14

For a 3×3 Matrix

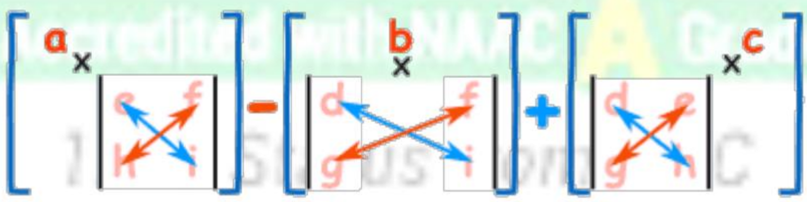
For a 3×3 matrix (3 rows and 3 columns):

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The determinant is:

|A| = a(ei − fh) − b(di − fg) + c(dh − eg)
*"The determinant of A equals ... etc"*

It may look complicated, but there is a pattern:




To work out the determinant of a 3×3 matrix:

- Multiply a by the determinant of the 2×2 matrix that is not in a's row or column.
- Likewise for b, and for c
- Sum them up, but remember the minus in front of the b

As a formula *(remember the vertical bars || mean "determinant of")*:

$$|A| = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

*"The determinant of A equals a times the determinant of ... etc"*

Example:

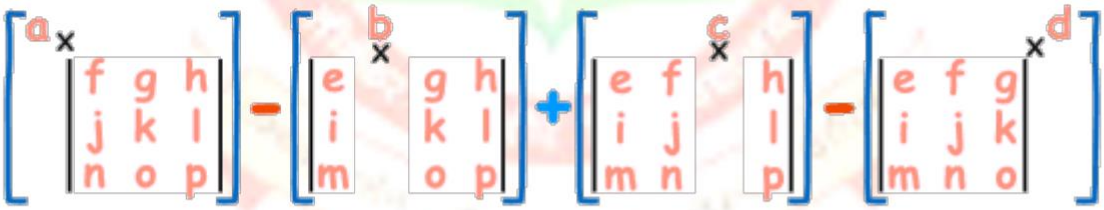
$$C = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$$

|C|= 6×(−2×7 )−(5×8) − 1×(4×7)−(5×2) + 1×(4×8) − (−2×2)
= 6×(−54) − 1×(18) + 1×(36)
= −306

For 4×4 Matrices and Higher

The pattern continues for 4×4 matrices:

- plus a times the determinant of the matrix that is not in a's row or column,
- minus b times the determinant of the matrix that is not in b's row or column,
- plus c times the determinant of the matrix that is not in c's row or column,
- minus d times the determinant of the matrix that is not in d's row or column,



As a formula:

$$|A| = a \cdot \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b \cdot \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c \cdot \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d \cdot \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Notice the +−+− pattern (+a... −b... +c... −d...). This is important to remember.
The pattern continues for 5×5 matrices and higher.

Important Properties of Determinants

**1. Reflection Property:**

**The determinant remains unaltered if its rows are changed into columns and the columns into rows. This is known as the property of reflection.**

**2. All-zero Property:**

If all the elements of a row (or column) are zero, then the determinant is zero.

**3. Proportionality (Repetition) Property:**

If the all elements of a row (or column) are proportional (identical) to the elements of some other row (or column), then the determinant is zero.

**4. Switching Property:**

The interchange of any two rows (or columns) of the determinant changes its sign.

**5. Scalar Multiple Property:**

If all the elements of a row (or column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant.

**6. Sum Property:**

**7. Property of Invariance:**

That is, a determinant remains unaltered under an operation of the form

$C_i \rightarrow C_i + \alpha C_j + \beta C_k$,**8. Factor Property:**

If a determinant $\Delta$ becomes zero when we put $x = \alpha$,

then $(x - \alpha)$ is a factor of $\Delta$.

**9. Triangle Property:**

If all the elements of a determinant above or below the main diagonal consist of zeros, then the determinant is equal to the product of diagonal elements. That is,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3$$

**10. Determinant of cofactor matrix:**

Δ2whereCijdenotesthecofactoroftheelementaijinΔ.

**Example Problems on Properties of Determinants**

**Question 1:** Using properties of determinants, prove that**Solution:**

By using invariance and scalar multiple property of determinant we can prove the given problem.

$$= (a + b + c)\,[(c - b)(b - c) - (a - b)(a - c)]$$

$$= (a+b+c)\left(ab+bc+ca-a^2-b^2-c^2\right)$$

*Takingα,β,γcommonfromC*1,*C*2,*C*3*respectively*Δ=*Nowtaking*[*α*,*β*,*γ*]*commonfromR*1,*R*2,*R*3*respectively*

**PROPERTIESOFDETERMINANTS**
**Here is the same list of properties that is contained the previous lecture.**

**(1.) A multiple of one row of "A" is added to another row to produce a matrix, "B", then:**

.

**(2.) If two rows are interchanged to produce a matrix, "B", then:** $|\mathbf{B}| = -|\mathbf{A}|$.

**(3.) If one row is multiplied by "k" to produce a matrix, "B", then:** $|\mathbf{B}| = \mathbf{k}\cdot|\mathbf{A}|$.

**(4.) If "A" and "B" are both n x n matrices then:** $|\mathbf{A}\cdot\mathbf{B}| = |\mathbf{A}|\cdot|\mathbf{B}|$. **(5.)** $|\mathbf{A}^{\mathbf{T}}| = |\mathbf{A}|$.

<u>**Example # 1**</u>**: Find the determinant by row reduction to echelon form.**

$$|\mathbf{A}| = \left|\begin{pmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{pmatrix}\right| \begin{pmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 3 & -3 \end{pmatrix} \begin{pmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{pmatrix}$$

**We now have "A" in upper triangular form. Since we have 3 pivots, "A" is invertible. If we continue the reduction process we could obtain a diagonal matrix.**

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$|\mathbf{A}| = \left|\begin{pmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{pmatrix}\right| = \left|\begin{pmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{pmatrix}\right| = \left|\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right|$$

$$\left|\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right| = (1)\cdot(1)\cdot(3)\cdot|\mathbf{I}| = 3$$

**Evidently, we only needed to go as far as echelon form to identify the values of the pivots, since the determinant is their product.**

**<u>Example # 2</u>: Find det B given det A.**

$$|\mathbf{A}| = \left|\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\right| = 7$$

$$|\mathbf{B}| = \left|\begin{pmatrix} a & b & c \\ 3\cdot g & 3\cdot h & 3\cdot i \\ d & e & f \end{pmatrix}\right|$$

$$|\mathbf{B}| = 3\cdot\left|\begin{pmatrix} a & b & c \\ g & h & i \\ d & e & f \end{pmatrix}\right| = (3)\cdot(-1)\cdot\left|\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\right| = (3)\cdot(-1)\cdot|\mathbf{A}|$$

$$|\mathbf{B}| = (3)\cdot(-1)\cdot|\mathbf{A}| = [(3)\cdot(-1)\cdot(7)] = -21$$

**<u>Example # 3</u>: Find det B given det A.**

$$|\mathbf{A}| = \left|\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\right| = 7$$

$$|\mathbf{B}| = \left|\begin{pmatrix} a & b & c \\ 2\cdot d + a & 2\cdot e + b & 2\cdot f + c \\ g & h & i \end{pmatrix}\right|$$

$$2\cdot |\mathbf{A}| = \left|\begin{pmatrix} a & b & c \\ 2\cdot d & 2\cdot e & 2\cdot f \\ g & h & i \end{pmatrix}\right| = \left|\begin{pmatrix} a & b & c \\ 2\cdot d + a & 2\cdot e + a & 2\cdot f + a \\ g & h & i \end{pmatrix}\right|$$

$$|\mathbf{B}| = 2\cdot |\mathbf{A}| = (2)\cdot(7) = 14$$

**<u>Example # 4</u>: Show that if 2 rows of a square matrix "A" are the same, then det A = 0.**

**Suppose rows "i" and "j" are identical. Then if we exchange those rows, we get the same matrix and thus the same determinant. However, a row exchange changes the sign of the determinant. This requires that $|\mathbf{A}| = -|\mathbf{A}|$, which can only be true if $|\mathbf{A}| = 0$.**

**<u>Example # 5</u>: Use determinants to decide if the set of vectors is linearly independent.**

$$\begin{pmatrix} 3 \\ 5 \\ -6 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 2 \\ -6 \\ 0 \\ 7 \end{pmatrix} \quad \begin{pmatrix} -2 \\ -1 \\ 3 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ -3 \end{pmatrix}$$



$$|\mathbf{A}| = \left|\begin{pmatrix} 3 & 2 & -2 & 0 \\ 5 & -6 & -1 & 0 \\ -6 & 0 & 3 & 0 \\ 4 & 7 & 0 & -3 \end{pmatrix}\right|$$

$$|\mathbf{A}| = |\mathbf{A}^{\mathbf{T}}| = \left|\begin{pmatrix} 3 & 5 & -6 & 4 \\ 2 & -6 & 0 & 7 \\ -2 & -1 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}\right| = \left|\begin{pmatrix} -1 & 3 & 0 & 4 \\ 2 & -6 & 0 & 7 \\ -2 & -1 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}\right|$$

$$\left|\begin{pmatrix} -1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 15 \\ 0 & -7 & 3 & -8 \\ 0 & 0 & 0 & -3 \end{pmatrix}\right| \quad \left|\begin{pmatrix} -1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 15 \\ 0 & -7 & 3 & -8 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right| = 0$$

**The vectors are Linearly Dependent.**

**Example # 6: Show that if "A" is invertible, then** $|\mathbf{A}^{-1}| = \dfrac{1}{|\mathbf{A}|}$.

$$\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{I}$$

$$|\mathbf{A} \cdot \mathbf{A}^{-1}| = |\mathbf{I}| = 1 = |\mathbf{A}| \cdot |\mathbf{A}^{-1}|$$

$$|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|}$$

**Example # 7: Show that if "A" is invertible, then** $|\mathbf{A}^{\mathbf{T}}| = |\mathbf{A}|$.

$$\mathbf{A} = \mathbf{L} \cdot \mathbf{U}$$

$$\mathbf{A}^{\mathbf{T}} = (\mathbf{L} \cdot \mathbf{U})^{\mathbf{T}} = \mathbf{U}^{\mathbf{T}} \cdot \mathbf{L}^{\mathbf{T}}$$

**The matrix "L" is lower triangular. Its transpose is upper triangular. The determinants of upper and lower non-singular matrices are the products of their diagonal elements. Since the transpose does not change the diagonal elements, then** $|\mathbf{L}^{\mathbf{T}}| = |\mathbf{L}|$ **and** $|\mathbf{U}^{\mathbf{T}}| = |\mathbf{U}|$.

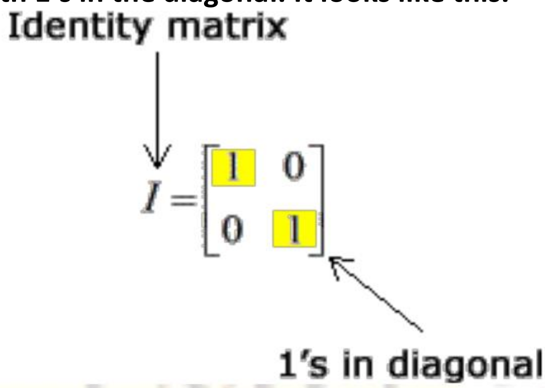
$$|\mathbf{A}^{\mathbf{T}}| = |\mathbf{U}^{\mathbf{T}}| \cdot |\mathbf{L}^{\mathbf{T}}| = |\mathbf{U}| \cdot |\mathbf{L}| = |\mathbf{A}|$$

**Properties of Determinants**

**Inverse of a 2×2 Matrix**

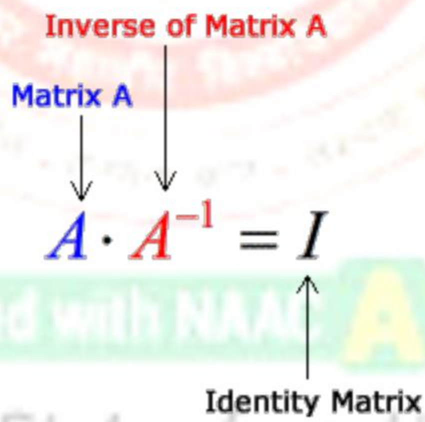
In this lesson, we are only going to deal with **2×2 square matrices. I have prepared five (5) worked examples to illustrate the procedure on how to solve or find the inverse matrix using the Formula Method.**

**Just to provide you with the general idea, two matrices are inverses of each other if their product is the identity matrix. An identity matrix with a dimension of 2×2 is a matrix with zeros everywhere but with 1's in the diagonal. It looks like this.**




**It is important to know how a matrix and its inverse are related by the result of their product. So then,**

**If a 2×2 matrix A is invertible and is multiplied by its inverse (denoted by the symbol $A^{-1}$), the resulting product is the Identity matrix which is denoted by** I*I*. To illustrate this concept, see the diagram below.




In fact, I can switch the order or direction of multiplication between matrices A and $A^{-1}$, and I would still get the Identity matrix I*I*. That means invertible matrices are commutative.

$$A^{-1} \cdot A = I$$

How do find the inverse of a matrix? The formula is rather simple. As long as you follow it, there shouldn't be any problem. Here go.

The Formula to Find the Inverse of a 2×2 Matrix

Given the matrix A

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Its inverse is calculated using the formula

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Wheredet$A$ is read as the determinant of matrix A.
A few observations about the formula:

- Entries $a$ and $d$ from matrix A are swapped or interchanged in terms of position in the formula.
- Entries $b$ and $c$ from matrix A remain in their current positions, however, the signs are reversed. In other words, put negative symbols in front of entries b and c.
- Since det$A$ is just a number, then1 /det$A$ is also a number that would serve as the scalar multiplier to the matrix

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Examples of How to Find the Inverse of a 2×2 Matrix

**Example 1: Find the inverse of the 2×2 matrix below, if it exists.**

$$A = \begin{bmatrix} 5 & 2 \\ -7 & -3 \end{bmatrix}$$

**The formula requires us to find the determinant of the given matrix. Do you remember how to do that? If not, that's okay. Review the formula below how to solve for the determinant of a 2×2 matrix.**

$$\text{Let matrix } A \text{ be } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } \det A = ad - bc$$

Always subtract!

**So then, the** determinant **of matrix A is**

$$\det A = \det\begin{bmatrix} 5 & 2 \\ -7 & -3 \end{bmatrix} = (5)(-3)-(2)(-7)$$
$$= (-15)-(-14)$$
$$= -15+14$$
$$\det A = -1$$

**To find the inverse, I just need to substitute the value of det*A*=−1 into the formula and perform some “reorganization” of the entries, and finally, perform scalar multiplication.**

- **Here goes again the formula to find the inverse of a 2×2 matrix.**

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**Now, let’s find the inverse of matrix A.**

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} -3 & -2 \\ 7 & 5 \end{bmatrix}$$

$$= -1\begin{bmatrix} -3 & -2 \\ 7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)(-3) & (-1)(-2) \\ (-1)(7) & (-1)(5) \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & 2 \\ -7 & -5 \end{bmatrix}$$

Let’s then check if our inverse matrix is correct by performing matrix multiplication of **A** and **A$^{-1}$ in two ways, and see if we’re getting the Identity matrix.**

$$AA^{-1} = \begin{bmatrix} 5 & 2 \\ -7 & -3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -7 & -5 \end{bmatrix} = \begin{bmatrix} (5)(3)+(2)(-7) & (5)(2)+(2)(-5) \\ (-7)(3)+(-3)(-7) & (-7)(2)+(-3)(-5) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A=\begin{bmatrix}3 & 2\\ -7 & -5\end{bmatrix}\begin{bmatrix}5 & 2\\ -7 & -3\end{bmatrix}=\begin{bmatrix}(3)(5)+(2)(-7) & (3)(2)+(2)(-3)\\ (-7)(5)+(-5)(-7) & (-7)(2)+(-5)(-3)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

**Since multiplying both ways generate the Identity matrix, then we are guaranteed that the inverse matrix obtained using the formula is the correct answer**

**Example 2: Find the inverse of the 2×2 matrix below, if it exists.**

$$B=\begin{bmatrix}-3 & 1\\ 5 & -2\end{bmatrix}$$

**First, find the** determinant **of matrix B.**

$$\det B=\det\begin{bmatrix}-3 & 1\\ 5 & -2\end{bmatrix}=(-3)(-2)-(1)(5)$$
$$=(6)-(5)$$
$$\det B=1$$

**Secondly, substitute the value of det B = 1 into the formula, and then reorganize the entries of matrix B to conform with the formula.**

$$B^{-1}=\frac{1}{\det B}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$
$$=\frac{1}{1}\begin{bmatrix}-2 & -1\\ -5 & -3\end{bmatrix}$$
$$=1\begin{bmatrix}-2 & -1\\ -5 & -3\end{bmatrix}$$
$$=\begin{bmatrix}(1)(-2) & (1)(-1)\\ (1)(-5) & (1)(-3)\end{bmatrix}$$
$$B^{-1}=\begin{bmatrix}-2 & -1\\ -5 & -3\end{bmatrix}$$

$$BB^{-1}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=B^{-1}B$$

**In other words, the matrix product of B and $B^{-1}$ in either direction yields the Identity matrix.**

**Example 3: Find the inverse of the matrix below, if it exists.**

$$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

**This is a great example because the determinant is neither** +1+1 nor −1−1 which usually results in an inverse matrix having rational or fractional entries. I must admit that the majority of problems given by teachers to students about the inverse of a 2×2 matrix is similar to this.

**Step 1**: Find the determinant of matrix C.

- The formula to find the determinant

$$\det C = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a)(d) - (b)(c)$$

Below is the animated solution to calculate the determinant of matrix C

$$\det C = \det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = (1)(4) - (2)(3)$$

$$= (4) - (6)$$

$$\det C = -2$$

**Step 2: The determinant of matrix C is equal to** −2−2. Plug the value in the formula then simplify to get the inverse of matrix C.

$$C^{-1} = \frac{1}{\det C}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{-2}\right)(4) & \left(\frac{1}{-2}\right)(-2) \\ \left(\frac{1}{-2}\right)(-3) & \left(\frac{1}{-2}\right)(1) \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$$

**Step 3: Check if the computed inverse matrix is correct by performing left and** right matrix multiplication **to get the Identity matrix.**

Always get the Identity matrix

$$CC^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = C^{-1}C$$

Multiply both ways

**matrix multiplication works in both cases as shown below.**

**First case:**

$$CC^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} (1)(-2)+(2)\left(\frac{3}{2}\right) & (1)(1)+(2)\left(\frac{-1}{2}\right) \\ (3)(-2)+(4)\left(\frac{3}{2}\right) & (3)(1)+(4)\left(\frac{-1}{2}\right) \end{bmatrix}$$

$$CC^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**Second case:**

$$C^{-1}C = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (-2)(1)+(1)(3) & (-2)(2)+(1)(4) \\ \left(\frac{3}{2}\right)(1)+\left(\frac{-1}{2}\right)(3) & \left(\frac{3}{2}\right)(2)+\left(\frac{-1}{2}\right)(4) \end{bmatrix}$$

$$C^{-1}C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**Example 4: Find the inverse of the matrix below, if it exists.**

$$D = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

**In our previous three examples, we were successful in finding the inverse of the given 2×2 matrices. I don't want to give you the impression that all2×2 matrices have inverses.**

**In this example, I want to illustrate when a given 2×2 matrix fails to have an inverse. How does that happen?**

**If review the formula again, it is obvious that this situation can occur when the determinant of the given matrix is zero because 1 divided by zero is undefined. And so, an undefined term distributed into each entry of the matrix does not make any sense.**

***This fraction becomes undefined!***

$$D^{-1} = \frac{1}{\det D} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**If det *D* = 0 ...**



**Let's go back to the problem to find the** determinant **of** matrix D.

$$\det D = \det \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$
$$= (4)(1) - (2)(2)$$
$$= 4 - 4$$
$$\det D = 0$$

Therefore, **the inverse of matrix D does not exist because the determinant of D equals zero.**

**Example 5: Find the inverse of the matrix below, if it exists.**

$$E = \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix}$$

**Step 1: Find the** determinant **of matrix E.**

$$\det E = \det \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix}$$
$$= (1)(4) - (-1)(3)$$
$$= (4) - (-3)$$
$$= 4 + 3$$
$$\det E = 7$$

**Step 2: Reorganize the entries of matrix E to conform with the formula, and substitute the solved value of the determinant of matrix E. Distribute the value of** 1 \ det*E*to the entries of matrix E then simplify, if possible.

$$E^{-1} = \frac{1}{\det E} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 4 & 1 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{7}\right)(4) & \left(\frac{1}{7}\right)(1) \\ \left(\frac{1}{7}\right)(-3) & \left(\frac{1}{7}\right)(1) \end{bmatrix}$$

$$E^{-1} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} \\ \frac{-3}{7} & \frac{1}{7} \end{bmatrix}$$

**Step 3: Verify your answer by checking that you get the Identity matrix in both scenarios.**

$$EE^{-1} = E^{-1}E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**First scenario:**

$$EE^{-1} = \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} \frac{4}{7} & \frac{1}{7} \\ \frac{-3}{7} & \frac{1}{7} \end{bmatrix}$$

$$= \begin{bmatrix} (1)\left(\frac{4}{7}\right) + (-1)\left(\frac{-3}{7}\right) & (1)\left(\frac{1}{7}\right) + (-1)\left(\frac{1}{7}\right) \\ (3)\left(\frac{4}{7}\right) + (4)\left(\frac{-3}{7}\right) & (3)\left(\frac{1}{7}\right) + (4)\left(\frac{1}{7}\right) \end{bmatrix}$$

$$EE^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**Second scenario:**

$$E^{-1}E = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} \\ \frac{-3}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{4}{7}\right)(1) + \left(\frac{1}{7}\right)(3) & \left(\frac{4}{7}\right)(-1) + \left(\frac{1}{7}\right)(4) \\ \left(\frac{-3}{7}\right)(1) + \left(\frac{1}{7}\right)(3) & \left(\frac{-3}{7}\right)(-1) + \left(\frac{1}{7}\right)(4) \end{bmatrix}$$

$$EE^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**Inverse of a Matrix**

**Please read our** Introduction to Matrices **first.**

What is the Inverse of a Matrix?

This is the reciprocal of a **number**:

Reciprocal of a Number

The **Inverse of a Matrix** is the **same idea** but we write it $A^{-1}$

Why not $^1/_A$ ? Because we don't divide by a matrix! And anyway $^1/_8$ can also be written $8^{-1}$

And there are other similarities:

When we **multiply a number** by its **reciprocal** we get **1**

$$8\times(^{1}/_{8})=\mathbf{1}$$

When we **multiply a matrix** by its **inverse** we get the **Identity Matrix** (which is like "1" for matrices):

$$A\times A^{-1}=\mathbf{I}$$

Same thing when the inverse comes first:

$$(^{1}/_{8})\times 8=\mathbf{1}$$
$$A^{-1}\times A=\mathbf{I}$$

**Identity Matrix**

We just mentioned the "Identity Matrix". It is the matrix equivalent of the number "1":

$$I=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

A 3x3 Identity Matrix

- It is "square" (has same number of rows as columns),
- It has **1**s on the diagonal and **0**s everywhere else.
- Its symbol is the capital letter **I**.

The Identity Matrix can be 2×2 in size, or 3×3, 4×4, etc ...



**Definition**

Here is the definition:

The inverse of A is $A^{-1}$ only when:

$$A\times A^{-1}=A^{-1}\times A=\mathbf{I}$$

Sometimes there is no inverse at all.

**2x2 Matrix**

OK, how do we calculate the inverse?

Well, for a 2x2 matrix the inverse is:

In other words: **swap** the positions of a and d, put **negatives** in front of b and c, and **divide** everything by the determinant (ad-bc).

Let us try an example:

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{4\text{x}6-7\text{x}2} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

How do we know this is the right answer?

Remember it must be true that: $A \times A^{-1} = \mathbf{I}$

So, let us check to see what happens when we multiply the matrix by its inverse:

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 4\times0.6 + 7\times-0.2 & 4\times-0.7 + 7\times0.4 \\ 2\times0.6 + 6\times-0.2 & 2\times-0.7 + 6\times0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 2.4 - 1.4 & -2.8 + 2.8 \\ 1.2 - 1.2 & -1.4 + 2.4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

And, hey!, we end up with the Identity Matrix! So it must be right.

It should **also** be true that: $A^{-1} \times A = \mathbf{I}$

Why don't you have a go at multiplying these? See if you also get the Identity Matrix:

$$\begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Why Do We Need an Inverse?

Because with matrices we **don't divide**! Seriously, there is no concept of dividing by a matrix.

But we can **multiply by an inverse**, which achieves the same thing.

Imagine we can't divide by numbers ...

... and someone asks "How do I share 10 apples with 2 people?"

But we can take the **reciprocal** of 2 (which is 0.5), so we answer:

$$\mathbf{10 \times 0.5 = 5}$$

They get 5 apples each.

The same thing can be done with matrices:

Say we want to find matrix X, and we know matrix A and B:

$$XA = B$$

It would be nice to divide both sides by A (to get X=B/A), but remember **we can't divide**.

But what if we multiply both sides by $A^{-1}$ ?

$$XA^{-1} = BA^{-1}$$

And we know that $AA^{-1} = I$, so:

$$XI = BA^{-1}$$

We can remove I (for the same reason we can remove "1" from 1x = ab for numbers):

$$X = BA^{-1}$$

And we have our answer (assuming we can calculate $A^{-1}$)

In that example we were very careful to get the multiplications correct, because with matrices the order of multiplication matters. AB is almost never equal to B

# Using matrices when solving system of equations

Matrices could be used to solve systems of equations but first one must master to find the inverse of a matrice, $C^{-1}$.

A matrices C will have an inverse $C^{-1}$ if and only if the determinant of C is not equal to zero.

$$if\, c = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

$$then\, C^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We will now in an example show how to solve systems of equations using matrices and the inverse of matrices.

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**Example**

Consider the following simultaneous equations (this example is also shown in our video lesson)

$$\begin{cases} 3x + y = 5 \\ 2x - y = 0 \end{cases}$$

Provided that we know how to multiply matrices we realize that our equations could be written as

Our solution is (1,2), the easiest way to check if we are right is to plug our values into our original equations.

***Cramer's Rule***

Given a system of linear equations, Cramer's Rule is a handy way to solve for just one of the variables without having to solve the whole system of equations. They don't usually teach Cramer's Rule this way, but this is supposed to be the point of the Rule: instead of solving the entire system of equations, you can use Cramer's to solve for just one single variable.

Let's use the following system of equations:

$$\begin{aligned} 2x + y + z &= 3 \\ x - y - z &= 0 \\ x + 2y + z &= 0 \end{aligned}$$

We have the left-hand side of the system with the variables (the "coefficient matrix") and the right-hand side with the answer values. Let $D$ be the determinant of the coefficient matrix of the above system, and let $D_x$ be the determinant formed by replacing the $x$-column values with the answer-column values:

Evaluating each determinant (using the method explained here), get:

$$D = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = (-2)+(-1)+(2) - (-1)-(-4)-(1) = 3$$

$$D_x = \begin{vmatrix} 3 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \end{vmatrix} = (-3)+(0)+(0) - (0)-(-6)-(0) = -3+6 = 3$$

$$D_y = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} = (0)+(-3)+(0) - (0)-(0)-(3) = -3-3 = -6$$

$$D_z = \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = (0)+(0)+(6) - (-3)-(0)-(0) = 6+3 = 9$$



Cramer's Rule says that $x = D_x \div D$, $y = D_y \div D$, and $z = D_z \div D$. That is:

$$x = {}^{3}/_{3} = 1,\ y = {}^{-6}/_{3} = -2, \text{ and } z = {}^{9}/_{3} = 3$$

That's all there is to Cramer's Rule. To find whichever variable you want (call it "ß" or "beta"), just evaluate the determinant quotient $D_ß \div D$. (Please don't ask me to explain why this works. Just trust me that determinants can work many kinds of magic.)

- **Given the following system of equations, find the value of *z*.**

$$2x + y + z = 1$$
$$x - y + 4z = 0$$
$$x + 2y - 2z = 3$$

To solve only for *z*, I first find the coefficient determinant.

$$D = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 4 \\ 1 & 2 & -2 \end{vmatrix} = (4) + (4) + (2) - (-1) - (16) - (-2) = 10 - 13 = -3$$

Then I form $D_z$ by replacing the third column of values with the answer column:

$$D_z = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 3 \end{vmatrix} = (-6) + (0) + (2) - (-1) - (0) - (3) = -4 - 2 = -6$$

Then I form the quotient and simplify: $\frac{D_z}{D} = \frac{-6}{-3} = 2$

***z* = 2**

The point of Cramer's Rule is that you don't have to solve the whole system to get the one value you need. This saved me a fair amount of time on some physics tests. I forget what we were working on (something with wires and currents, I think), but Cramer's Rule was so much faster than any other solution method (and God knows I needed the extra time). Don't let all the subscripts and stuff confuse you; the Rule is really pretty simple. You just pick the variable you want to solve for, replace that variable's column of values in the coefficient determinant with the answer-column's values, evaluate that determinant, and divide by the coefficient determinant. That's all there is to it.

**Almost**

What if the coefficient determinant is zero? You can't divide by zero, so what does this mean? I can't go into the technicalities here, but "$D = 0$" means that the system of equations has no unique solution. The system may be inconsistent (no solution at all) or dependent (an infinite solution, which may be expressed as a parametric solution such as "($a$, $a + 3$, $a - 4$)"). In terms of Cramer's Rule, "$D = 0$" means that you'll have to use some other method (such as matrix row operations) to solve the system. If $D = 0$, you can't use Cramer's Rule.

## UNIT – V
## APPLICATIONS OF MATRICES IN INPUT - OUTPUT ANALYSIS

Input-output is a novel technique invented by Professor Wassily W. Leontief in 1951. It is used to analyse inter-industry relationship in order to understand the inter-dependencies and complexities of the economy and thus the conditions for maintaining equilibrium between supply and demand.

Thus it is a technique to explain the general equilibrium of the economy. It is also known as "inter-industry analysis". Before analysing the input-output method, let us understand the meaning of the terms, "input" and "output". According to Professor J.R. Hicks, an input is "something which is bought for the enterprise" while an output is "something which is sold by it."

An input is obtained but an output is produced. Thus input represents the expenditure of the firm, and output its receipts. The sum of the money values of inputs is the total cost of a firm and the sum of the money values of the output is its total revenue.

The input-output analysis tells us that there are industrial interrelationships and inter-dependencies in the economic system as a whole. The inputs of one industry are the outputs of another industry and vice versa, so that ultimately their mutual relationships lead to equilibrium between supply and demand in the economy as a whole.

Coal is an input for steel industry and steel is an input for coal industry, though both are the outputs of their respective industries. A major part of economic activity consists in producing intermediate goods (inputs) for further use in producing final goods (outputs).

There are flows of goods in "whirlpools and cross currents" between different industries. The supply side consists of large inter-industry flows of intermediate products and the demand side of the final goods. In essence, the input-output analysis implies that in equilibrium, the money value of aggregate output of the whole economy must equal the sum of the money values of inter-industry inputs and the sum of the money values of inter-industry outputs.

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1. **Main Features**

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**1. Main Features:**

**The input-output analysis is the finest variant of general equilibrium. As such, it has three main elements; Firstly, the input-output analysis concentrates on an economy which is in equilibrium. Secondly, it does not concern itself with the demand analysis. It deals exclusively with technical problems of production. Lastly, it is based on empirical investigation. The input-output analysis consists of two parts: the construction of the input-output table and the use of input-output model.**

**2. The Static Input-Output Model:**

**The input-output model relates to the economy as a whole in a particular year. It shows the values of the flows of goods and services between different productive sectors especially inter-industry flows.**

**Assumptions:**

**This analysis is based on the following assumptions:**

**(i) The whole economy is divided into two sectors—"inter-industry sectors" and "final-demand sectors," both being capable of sub-sectoral division.**

**(ii) The total output of any inter-industry sector is generally capable of being used as inputs by other inter-industry sectors, by itself and by final demand sectors.**

**(iii) No two products are produced jointly. Each industry produces only one homogeneous product.**

**(iv) Prices, consumer demands and factor supplies are given.**

**(v) There are constant returns to scale.**

**(vi) There are no external economies and diseconomies of production.**

**(vii) The combinations of inputs are employed in rigidly fixed proportions. The inputs remain in constant proportion to the level of output. It implies that there is no substitution between different materials and no technological progress. There are fixed input coefficients of production.**

**Explanation:**

**For understanding, a three-sector economy is taken in which there are two inter-industry sec-tors, agriculture and industry, and one final demand sector.**

**Table 1 provides a simplified picture of such economy in which the total output of the industrial, agricultural and household sectors is set in rows (to be read horizontally) and has been divided into the agricultural, industrial and final demand sectors. The inputs of these sectors are set in columns. The first row total shows that altogether the agricultural output is valued at Rs. 300 crores per year.**

**Table 1 : Input-Output Table**

*(In value terms) (Rs. Crores)*

| | | Purchasing Sectors | | | |
|---|---|---|---|---|---|
| | *Sectors* | *1 Inputs to Agriculture* | *2 Inputs to Industry* | *3 Final Demand* | *Total Output or Total Revenue* |
| Selling Sectors | Agriculture | 50 | 150 | 100 | 300 |
| | Industry | 100 | 250 | 150 | 500 |
| | Value added* | 150 | 100 | 0 | 250 |
| | Total input or Total Cost | 300 | 500 | 250 | 1050 |

**Of this total, Rs. 100 crores go directly to final consumption (demand), that is, household and government, as shown in the third column of the first row. The remaining output from agriculture goes as inputs: 50 to itself and 150 to industry. Similarly, the second row shows the distribution of total output of the industrial sector valued at Rs. 500 crores per year. Columns 1, 2 and 3 show that 100 units of manufactured goods go as inputs to agriculture, 250 to industry itself and 150 for final consumption to the household sector.**

Let us take the columns (to be read downwards). The first column describes the input or cost structure of the agricultural industry. Agricultural output valued at Rs. 300 crores is produced with the use of agricultural goods worth Rs. 50, manufactured goods worth Rs. 100 and labour or/and management services valued at Rs. 150. To put it differently, it costs Rs. 300 crores to get revenue of Rs. 300 crores from the agricultural sector. Similarly, the second column explains the input structure of the industrial sector (i.e., 150 + 250 + 100 = 500).

Thus "a column gives one point on the production function of the corresponding industry." The 'final demand' column shows what is available for consumption and government expenditure. The third row corresponding to this column has been shown as zero. This means that the household sector is simply a spending (consuming) sector that does not sell anything to itself. In other words, labour is not directly consumed.

There are two types of relationships which indicate and determine the manner in which an economy behaves and assumes a certain pattern of flows of resources.

They are:

(a) The internal stability or balance of each sector of the economy, and

(b) The external stability of each sector or intersectoral relationships. Professor Leontief calls them the "fundamental relationships of balance and structure." When expressed mathematically they are known as the "balance equations' and the "structural equations".

If the total output of say X. of the 'ith' industry is divided into various numbers of industries 1, 2, 3, n, then we have the balance equation:

$X_1 = x_{i1} + x_{i2} + x_{i3} + x_{in}...... + D_1$

and if the amount say У. absorbed by the "outside sector" is also taken into consideration, the balance equation of the $i^{th}$ industry becomes

$$X_i = x_{i1} + x_{i2} + x_{i3} + .......x_{in} + D_i + Y_i$$

or $$\sum_{j=1}^{n} x_{ij} + Y_i = X_i$$

**It is to be noted that $Y_i$ stands for the sum of the flows of the products of the ith industry to consumption, investment and exports net of imports, etc. It is also called the "final bill of goods" which it is the function of the output to fill. The balance equation shows the conditions of equilibrium between demand and supply. It shows the flows of outputs and inputs to and from one industry to other industries and vice versa.**

**Since $x_{12}$ stands for the amount absorbed by industry 2 of the ith industry, it follows that xij stands for the amount absorbed by the ith industry of jth industry.**

**The "technical coefficient" or "input coefficient" of the ith industry is denoted by:**

**aij = xij/Xj**

**where xij is the flow from industry i to industry j, Xj is the total output of industry aij and aij, as already noted above, is a constant, called "technical coefficient" or "flow coefficient" in the ith industry. The technical coefficient shows the number of units of one industry's output that are required to produce one unit to another industry's output.**

**Equation (3) is called a "structural equation." The structural equation tells us that the output of one industry is absorbed by all industries so that the flow structure of the entire economy is revealed. A number of structural equations give a summary description of the economy's existing technological conditions.**

**Using equation (3) to calculate the aij for our example of the two-sector input-output Table 1, we get the following technology matrix.**

**Table 2: Technology Coefficient Matrix A**

| | *Agriculture* | *Industry* |
|---|---|---|
| Agriculture | 50/300= .17 | 150/500= .30 |
| Industry | 100/300= .33 | 250/500= .50 |

**These input coefficients have been arrived at by dividing each item in the first column of Table 1 by first row total, and each item in the second column by the second row, and so on. Each column of the technological matrix reveals how much agricultural and industrial sectors require from each other to produce a rupee's worth of output. The first column shows that a**

rupee's worth of agricultural output requires inputs worth 33 paise from industries and worth 17 paise from agriculture itself.

**The Leontief Solution:**

The table can be utilised to measure the direct and indirect effects on the entire economy of any sectoral change in total output of final demand.

Again using equation (3)

aij = xij/Xj

Cross multiplying, xij = aij. Xj

By substituting the value of xij into equation (2) and transposing terms, we obtain the basic input- output system of equations

$$X_i - \sum_{i=1}^{n} aij\ xj = Yi$$

In terms of our two-sector economy, there would be two linear equations that could be written symbolically as follows:

$$x_1 - a_{11}x_1 - a_{12}x_2 = Y_1$$
$$x_2 - a_{21}x_1 - a_{22}x_2 = Y_2$$

The above symbolic relationship can be shown in matrix form:

X- [A]X= Y

X [I-A] = Y

where matrix (I – A) is known as the Leontief Matrix

$$(I-A)^{-1}(I-A)X = (I-A)^{-1}Y$$
$$X = (I-A)^{-1}Y \qquad [\because (I-A)^{-1}(I-A)]$$

and $I$, the identity matrix $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\text{Hence} \quad \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - [A] \right\}^{-1} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

**Numerical Solution:**

**Our technology matrix as per Table 2 is**

$$A = \begin{bmatrix} .1 & .3 \\ .3 & .5 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 100 \\ 150 \end{bmatrix}$$

$$(I - A) = \begin{bmatrix} .9 - .3 \\ .3 - .5 \end{bmatrix}$$

$$\text{The value of inverse} = \frac{\cdot \text{ Adjoint}}{\text{Determinant}} = \frac{Adj}{|A|}$$

$$[Aij] = \begin{bmatrix} .5 & .3 \\ .3 & .9 \end{bmatrix}$$

$$\text{By transposing, } Aij = \begin{bmatrix} .5 & .3 \\ .3 & .9 \end{bmatrix}$$

$$\text{The value of determinant} = .9(.5) - (-.3)(-.3)$$
$$= .45 - .09 = .36$$

$$\text{Hence } \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \frac{1}{= .36} \begin{bmatrix} .5 & .3 \\ .3 & .9 \end{bmatrix} \begin{bmatrix} 100 \\ 150 \end{bmatrix}$$

The total ŏutput of agriculture sector ($x_1$)

$$= \frac{.5 \times 100 + .3 \times 150}{.36} = 264$$

The total output of industrial sector ($x_2$)

$$= \frac{.3 \times 100 + .9 \times 150}{.36} = 458.$$

3. The Dynamic Input-output Model:

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So far we have studied an open static model. "The model becomes Dynamic when it is closed by the linking of the investment part of the final bill of goods to output. The dynamic input-output model extends the concept of inter-sectoral balancing at a given point of time to that of inter-sectoral balancing over time.

This necessarily involves the concept of durable capital. The Leontief dynamic input-output model is r generalization of the static model and is based on the same assumptions. In a dynamic model, the output of a given period is supposed to go into stocks,

i. e., capital goods, and the stocks, in turn, are distributed among industries.

The balance equation is:

$$X_i(t) = x_{i1}(t) + x_{i2}(t) + x_{i3}(t) \dots + x_{in}(t) + (S'_{i1} + S'_{i2} + S'_{i3} + \dots S'_{in}) + D_i(t) + Y_i(t)$$

Here $X_i$ (t) represents the total flow of output of ith industry in period t, which is used for three purposes:

(i) For production in the economy's n industries $x_{11}$ (t), $x_{12}$ (t), etc., in that period;

(ii) As net addition to the stock of capital goods in n industries i.e. $S'_t$ which can also be written as $\Delta S_1$ (t) = $S_1$ (t+1) – $S_1$ (t), where $S_1$ (t) indicates the accumulated stock of capital in the current period (t), and $S_1$ (t + l) is next year's stock; and

(iii) As consumption demand for the next period D. (t + 1). If we ignore depreciation and wear-tear, then S. (t+1) — $S_1$ (t) is the net addition to capital stock out of current production.

Equation (4) can, therefore, be written as:

$X_i$ (t) t $x_{1i1}$ + $x_{i2}$ + $x_{i3}$ + $x_{in}$ + S. (t +1) – $S_1$ (t) + $D_2$ (t) + Yi (t)

where $Y_i$ (t) stands for the amount absorbed by the outside sector in period t.

Just as the technical co-efficient was derived in the case of the static model, the capital co-efficient can be found out in a similar manner. Capital co-efficient of the ith product used by the jth industry is denoted by

bij =Sij /Xj

Cross multiplying, we have Sij = bij. X

where Sij represents the amount of capital stock of the ith product used by the jth industry. Xj is total output of industry j, and bij is a constant called capital co-efficient or stock co-efficient. Equation (5) is known as the structural equation in a dynamic model.

If the bij co-efficient is zero, it means that no stock is required by an industry and the dynamic model becomes a static model. Moreover, bij can neither be negative nor infinite. If the capital coefficient is negative, the input is, in fact, an output of an industry.

Input–Output Analysis

The input-output method is an adaptation of the neoclassical theory of general equilibrium to the empirical study of the quantitative interdependence between interrelated economic activities. It was originally developed to analyze and measure the connections between the various producing and consuming sectors within a national economy, but it has also been applied to the study of smaller economic systems, such as metropolitan areas or even large integrated individual enterprises, and to the analysis of international economic relationships.

In all instances the approach is basically the same: The interdependence of the individual sectors of the given system is described by a set of linear equations. The specific structural characteristics of the system are thus determined by the numerical magnitude of the coefficients of these equations. These coefficients must be determined empirically; in the analysis of the structural characteristics of an entire national economy, they are usually derived from a so-called statistical input-output table.

Applications. Application of the input-output method in empirical research requires the availability of basic statistical information. By 1963, input-output tables had been compiled for more than forty countries. The principal economic applications, as distinct from engineering and business-management applications, have been made in such fields as economic projections of demand, output, employment, and investment for the individual sectors of entire countries and of smaller economic regions (for example, metropolitan areas); study of technological change and its effect on productivity; analysis of the effect of wage, profit, and tax changes on prices; and study of international and interregional economic relationships, utilization of natural resources, and developmental planning.

Some of these applications require construction of special purpose input-output models. A great variety of special models is used, for instance, in the analysis of interregional relationships and in the study of problems of developmental planning.

Input-output tables

An input-output table describes the flow of goods and services between all the individual sectors of a national economy over a stated period of time—say, a year. An example of an input-output table depicting a three-sector economy is shown in Table 1. The three sectors are agriculture, whose total annual output amounted to 100 bushels of wheat; manufacturing, which produced 50 yards of cloth; and households, which supplied 300 man-years of labor. The nine entries inside the main body of the table show the intersectoral flows. Of the 100 bushels of wheat turned out by agriculture, 25 bushels were used up within the agricultural sector itself, 20 were delivered to and absorbed, as one of its inputs, by manufacturing, and 55 were taken by the household sector. The second and the third rows of the table describe in the same way the allocation of outputs of the two other sectors.

The figures entered in each column of the main body of the table thus describe the input structure

Table 5 - Examlpe of an input-output table (in physical units)

| Into From | Sector Agriculture | 1: Sector Manufacturing | 2: Sector Households | 3: Total output |
|---|---|---|---|---|
| Sector Agriculture | 1: 25 | 20 | 55 | 100 bushels of wheat |
| Sector Manufacturing | 2: 14 | 6 | 30 | 50 yards of cloth |
| Sector Households | 3: 80 | 180 | 40 | 300 man-years of labor |

of the corresponding sector. In producing 100 bushels of wheat, agriculture absorbed 25 bushels of its own products, 14 yards of manufactured goods, and 80 man-years of labor received from the households. In producing 50 yards of cloth, manufacturing absorbed 20 bushels of wheat, 6 yards of its own products, and 180 man-years of labor. In their turn, the households used their income, which they received for supplying 300 man-years of labor, to pay for 55 bushels of wheat, 30 yards of cloth, and 40 man-years of direct services of labor, which they consumed.

All entries in this table are supposed to represent quantities, or at least physical indexes of quantities, of specific goods or services. A less aggregative, more detailed input-output table describing the same national economy in terms of 50, 100, or even 1,000 different sectors would permit a more specific qualitative identification of the individual entries. In a larger table, manufacturing would be represented not by one but by many distinct industrial sectors; its output—and consequently also the inputs of the other sectors—would be described in terms of yards of cotton cloth and tons of paper products, or possibly yards of percale, yards of heavy cotton cloth, tons of newsprint, and tons of writing paper.

Input-output tables and income accounts

Although in principle the intersectoral flows, as represented in an input-output table, can be thought of as being measured in physical units, in practice most input-output tables are constructed in value terms. Table 2 represents a translation of Table 1 into value terms on the assumption that the price of wheat is $2 per bushel, the price of cloth is $5 per yard, and the price of services supplied by the household sector is $1 per man-year. Thus, the values of the total outputs of agriculture, manufacturing, and households are shown in Table 2 as $200 (= 100 x $2), $250 (= 50 x $5). and $300 (=300x$l), respectively. The last row shows the combined value of all outputs absorbed by each

Table 2 - Examlpe of an input-output table (in dollars)

| Into From | Sector Agriculture | 1: Sector Manufacturing | 2: Sector Households | 3: Total output |
|---|---|---|---|---|
| Sector Agriculture | 1: 50 | 40 | 110 | 200 |
| Sector Manufacturing | 2: 70 | 30 | 150 | 250 |
| Sector Households | 3: 80 | 180 | 40 | 300 |
| Total input | 200 | 250 | 300 | |

of the three sectors. Such column totals could not have been shown on Table 1, since the physical quantities of different inputs absorbed by each sector cannot be meaningfully added.

The input-output table expressed in value terms can be interpreted as a system of national accounts. The $300 showing the value of services rendered by households during the year obviously represents the annual national income. It equals the total of the income payments (shown in the third row) received by households for services rendered to each sector; it also equals the total value of goods and services (shown in the third column) purchased by households from themselves and from the other sectors. To the extent that the column entries (showing the input structure of each productive sector) cover current expenditures but not purchases made on capital account, the capital expenditures—being paid out of the net income—should be entered in the households' column.

All figures in Table 2, except the column sums shown in the bottom row, can also be interpreted as physical quantities of the goods or services to which they refer. This requires only that the physical unit in which one measures the entries in each row be redefined as the amount of output of the particular sector that can be purchased for $1 at prices that prevailed during the interval of time for which the table was constructed.

Input coefficients

Let the national economy be subdivided into $n + 1$ sectors. Sectors 1,..., $n$ are industries—that is, producing sectors—and sector $n + 1$ is the final demand sector, represented in input-output Tables 1 and 2 by households. For purposes of mathematical manipulation, the physical output of sector $i$ is usually represented by $x_i$, and the symbol $x_{ij}$ stands for the amount of the product of sector $i$ absorbed as an input by sector $j$. The quantity of the product of sector $i$ delivered to the final demand sector, $x_{i.n+1}$, is usually identified in short as $y_i$

The quantity of the output of sector $i$ absorbed by sector $j$ per unit of j's total output is and represented by the symbol $a_{ij}$ is called the *input coefficient* of sector $i$ into sector $j$. Thus,

A complete set of the input coefficients of all sectors of a given economy arranged in the form of a rectangular table, corresponding to the input-output table of the same economy, is called the *structural matrix* of that economy. Table 3 presents the structural matrix of the economy whose flow

Table 3 - Structural matrix corresponding to the input-output table of Table 1

| From Into | Sector 1: Agriculture | Sector 2: Manufacturing | Sector 3: Households |
|---|---|---|---|
| Sector 1: Agriculture | 0.25 | 0.40 | 0.183 |
| Sector 2: Manufacturing | 0.14 | 0.12 | 0.100 |
| Sector 3: Households | 0.80 | 3.60 | 0.133 |

matrix is shown in Table 1. The flow matrix constitutes the usual, although not necessarily the only possible, source of empirical information on the input structure of the various sectors of an economy. The entries in Table 3 are computed, according to equations (1), from the figures presented in Table 1—for example, $a_{11}$ = 25/100 = 0.25, and $a_{12}$ = 20/50 = 0.40.

In practice, the structural matrices are usually computed from input-output tables described in value terms, such as Table 2. In any case, the input coefficients must be interpreted, for analytical purposes described below, as ratios of two quantities measured in physical units. To emphasize this fact, we derived the structural matrix in this example from Table 1, not Table 2.

Theory of static input-output systems

The balance between the total output and the combined input uses of the product of each sector, as shown in tables 1 and 2, can be described by the following set of *n* equations:

A substitution of equations (1) into (2) yields *n* general equilibrium relationships between the total outputs, $x_1$, $x_2$,..., $x_n$, of the producing sectors and the final bill of goods, $y_1$, $y_2$,...,$y_n$, absorbed by households, government, and other final users:

If the final demands, $y_l$ $y_2$,..., $y_n$, that is, the quantities of the different goods absorbed by households and any other sector whose outputs are not represented by the variables appearing on the left-hand side of equations (3), are given, the system can be solved for the total outputs, $x_1$,..., $x_2$,..., $x_n$.

The general solution of these equilibrium equations for the unknown x's in terms of the given y's can be presented in the following form:

The constant $A_{ij}$ indicates by how much $x_i$ would increase if $y_j$ were increased by one unit. An increase in $y_i$ would affect sector *i* directly (and also indirectly) if $i = j$, but even if $i \neq j$, sector *i* is affected indirectly, since it has to provide additional inputs to all other sectors that must contribute directly or indirectly to producing the additional $y_{,i}$. From the computational point of view, this means that the magnitude of each coefficient $A_{ij}$- in the solution (4) depends, in general, on all the input coefficients appearing on the left-hand side of the system of equilibrium equations, (3). In mathematical language, the matrix

is the inverse of the matrix

The computation involved in finding the solution of (3) is called the *inversion of the coefficient matrix.* The inverse of the matrix

based on Table 3 is

(Each element of the inverse has been rounded to four decimal places.) When inserted into (4), this yields two equations—namely,

which permit us to determine the total outputs, $x_1$, and $x_2$, of agriculture and manufacturing corresponding to any given combination of the deliveries of their respective products, $y_1$ and $y_2$, to the exogenous household sector. For example, setting $y_1$=55 and $y_2$=30, we find that $x_1$,=100 and $x_2$=50, which agrees with the figures in Table 1. Only if all the $A_{ij}$ are nonnegative will there necessarily exist a set of *positive* total outputs for *any* given set of final deliveries. A sufficient condition for the nonnegativity of the $A_{ij}$ is that in the structural matrix

the sum of the coefficients in each column (or in each row) be not larger than one and that at least one of these column (or row) sums be smaller than one. A national economy whose structural matrix does not satisfy this condition will be unable to sustain itself—that is, the combined input requirements of all sectors in such an economy would exceed the combined productive capabilities of the sectors.

When the structural matrix of a national economy is derived from a set of empirically observed value flows, the condition stated above is generally found to be satisfied.

In applying this criterion to a given structural matrix, it is useful to keep in mind that by doubling the size of the physical unit used in measuring the output of a particular sector, one can double the magnitude of all the technical input coefficients that make up the corresponding row and reduce to one-half their previous size all entries in the corresponding column.

In an *open* input-output system, households are usually treated as an exogenous sector—that is, total output of households, $x_{n+1}$, which is total employment, usually does not appear as an unknown variable on the left-hand side of system (3) and on the right-hand side of the solution (4). After the outputs of the endogenous sectors have been determined, total employment can be computed from the following equation:

The technical coefficients, $a_{n+1,1}$, $a_{n+1,2}$, ..., $a_{n+1,n}$, are the inputs of labor absorbed by various industries (sectors) per unit of their respective outputs; j/"+, is the total amount of labor directly absorbed by households and other exogenous sectors. The employment equation for the three-sector system whose structural matrix is shown in Table 3 is

Households are not always treated as an exogenous sector. In dealing with problems of income generation in its relation to employment, the quantities of consumer goods and services absorbed by households can be considered (in a Keynesian manner) to be structurally dependent on the total level of employment, just as the quantities of coke and ore absorbed by blast furnaces are considered to be structurally related to the amount of pig iron produced by them. With households shifted to the left side of equations (2) and (4), the exogenous final demand appearing on the right side will contain only such items as government purchases, exports, and, in any case, additions to or reductions in stocks of goods—that is, real investment or disinvestment.

When all sectors and all purchases are considered to be endogenous, the input-output system is called *closed.* A static system cannot be truly closed since endogenous explanation of investment or disinvestment requires consideration of structural relationships between inputs and outputs that occur during different periods of time (see "Theory of dynamic input-output systems," below).

Exports and imports

In an input-output table of a country or a region that trades across its borders, exports can be entered as positive components and imports as negative components of final demand. If the economy described in Table 1 ceased to be self-sufficient and started, say, to import 20 bushels of wheat and to export 8 yards of cloth, while letting households consume the same amounts of both products as before, a new balance between all inputs and outputs would be established, which is described in Table 4.

The input coefficients of the endogenous sectors, and consequently also the structural matrix of the system and its inverse, remain the same as they were before. To form the new column of final demand, we have to add to the quantity of each good absorbed by households the amount that was exported less the amount that was imported. Defining $E_i$, $i=1,...n$, as net exports (exports minus imports) of good *i,* and redefining $x_{i,n+1}, i=1,...,n,$ as final demand for good *i* by households only, we have

The sectoral outputs can then be derived (see “Theory of static input-output systems,” above) from the general solution (4). For our numerical example, we can use equations (5) directly. The total labor requirement of the economy—300 man-years—remains in this particular case unchanged after the economy enters foreign trade, because the total direct and indirect labor content of the 20

ASSUMPTIONS

CONSTANT RETURNS TO SCALE

The same quantity of inputs is needed per unit of Output, regardless of the level of production. In other words, if Output increases by 10%, input requirements will also increase by 10%.

NO SUPPLY CONSTRAINTS

I-O assumes there are no restrictions to raw materials and employment and assumes there is enough to produce an unlimited amount of product. It is up to the user to decide whether this is a reasonable assumption for their study area and analysis, especially when dealing with large-scale impacts.

FIXED INPUT STRUCTURE

I-O assumes there are no restrictions to raw materials and employment and assumes there is enough to produce an unlimited amount of product. It is up to the user to decide whether this is a reasonable assumption for their study area and analysis, especially when dealing with large-scale impacts.

INDUSTRY TECHNOLOGY

By this assumption, each Industry's production requires a unique set of inputs, no matter which product it is producing. This assumption provides the basis for the mechanical calculation of the total requirements tables in the I-O accounts.

COMMODITY TECHNOLOGY

By this assumption, the production of each Commodity requires a unique set of inputs no matter which Industry produces it. This assumption provides the basis for the redefinition of secondary products in the I-O accounts, whereby the secondary product and its associated inputs are redefined from the Industry that produced it to the Industry in which it is the primary product.

CONSTANT MAKE MATRIX

As a requirement of the Industry technology assumption, Industry byproduct coefficients are constant. An Industry will always produce the same mix of Commodities regardless of the level of production. In other words, an Industry will not increase the Output of one product without proportionately increasing the Output of all its other products.

THE MODEL IS STATIC

No price changes are built in. The underlying data and relationships are not affected by impact runs. The relationships for a given year do not change unless another IMPLAN Data Year is purchased.

Input-Output (I-O) models do not account for general equilibrium effects such offsetting gains or losses in other sectors or geographies or the diversion of funds from other projects. I-O models assume that consumer preferences, government policy, technology, and prices all remain constant. By design, I-O models do not account for forward linkages.

It falls upon the analyst to take such possible countervailing or offsetting effects into account or to note the omission of such possible effects from the analysis. Price changes cannot be modeled in IMPLAN directly; instead, the final demand effects of a price change must be estimated by the analyst before modeling them in IMPLAN to estimate the additional economic impacts of such changes.